Townes soliton beyond mean field

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SciPost Phys. 19, 037 (2025)

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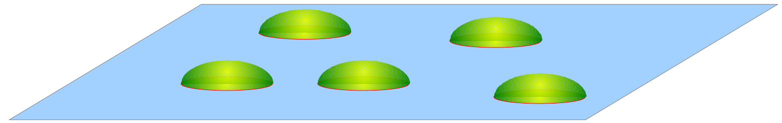
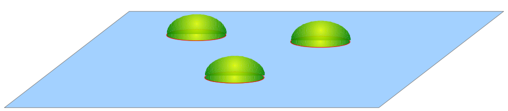
2D attractive bosons

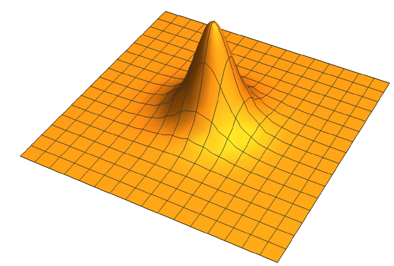


Bosons, dimension=2

zero-range attraction (a – unit of length or $B_2=-4e^{-2\gamma}/a^2$ - dimer energy)

single parameter: number of bosons N





N

N=3,4,...

Calculated ground-state energies, wave functions, and excited states

N=10-20

Monte-Carlo results for GS energies and predicted scaling

$$B_N \sim B_2 e^{4N/C}$$

N >> 1

Classical Townes soliton : optical and ultracold experiments

This work : beyond-mean-field (Bogoliubov) theory of the Townes soliton + dynamics (breathing mode)

Pitaevskii-Rosch scaling symmetry

and quantum anomaly

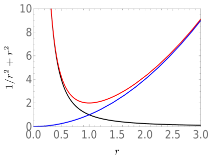
Classical (mean-field) Gross-Pitaevskii energy functional for 2D bosons (similar for 2D Fermi mixtures)

$$E_{\mathrm{MF}}(\Psi,\Psi^*) = (1/2)\int d^2\rho\,[|\nabla_\rho \Psi(\boldsymbol{\rho},t)|^2 + g|\Psi(\boldsymbol{\rho},t)|^4]$$

$$V(\boldsymbol{\rho}) = g\delta^2(\boldsymbol{\rho}) \quad : \quad V(\lambda\boldsymbol{\rho}) = \lambda^{-2}V(\boldsymbol{\rho})$$

dimensionless

$$V(r) = \beta/r^2$$ - other example (any dimension)

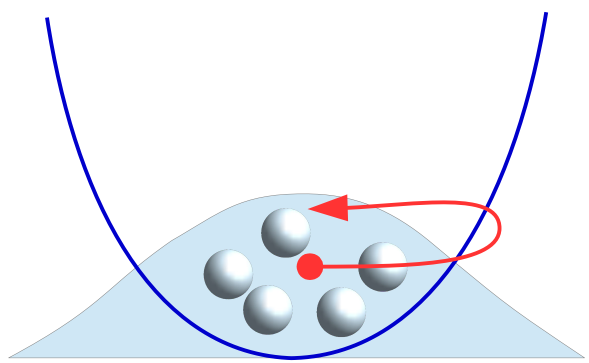


$$E_{\mathrm{MF}}(\Psi,\Psi^*) = (1/2)\int d^2\rho\,[|\nabla_\rho \Psi(\boldsymbol{\rho},t)|^2 + g|\Psi(\boldsymbol{\rho},t)|^4 + \omega^2\rho^2|\Psi(\boldsymbol{\rho},t)|^2]$$

⟹ Undamped breathing mode with frequency 2ω independent of interaction

[Pitaevskii'1996;Pitaevskii&Rosch'1997]

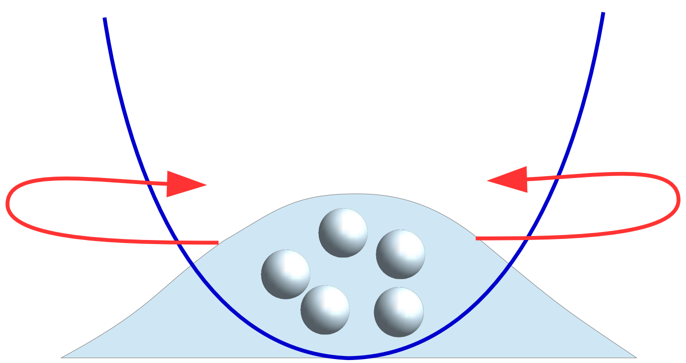
Any harmonic confinement and any interaction



Separation of center-of-mass from relative coordinates

Undamped dipolar mode with 1 X (trapping frequency)

Isotropic harmonic confinement and scale-invariant interaction: ideal gas, $1/r^2$ interaction, unitary 3D gas, 1D Tonks gas, 2D gases in mean-field limit



Separation of hyperradius (~rms size) from all other coordinates

Undamped breathing mode with 2 X (trapping frequency)

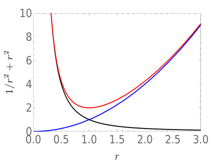
Classical (mean-field) Gross-Pitaevskii energy functional for 2D bosons (similar for 2D Fermi mixtures)

$$E_{\mathrm{MF}}(\Psi,\Psi^*) = (1/2)\int d^2\rho\,[|\nabla_\rho \Psi(\rho,t)|^2 + g|\Psi(\rho,t)|^4]$$

$$V(\rho) = g\delta^2(\rho) \quad : \quad V(\lambda\rho) = \lambda^{-2} V(\rho)$$

dimensionless

$$V(r) = \beta / r^2$$ - other example (any dimension)



$$E_{\mathrm{MF}}(\Psi,\Psi^*) = (1/2)\int d^2\rho\,[|\nabla_\rho \Psi(\rho,t)|^2 + g|\Psi(\rho,t)|^4 + \omega^2\rho^2|\Psi(\rho,t)|^2]$$

Undamped breathing mode with frequency 2ω independent of interaction

[Pitaevskii'1996;Pitaevskii&Rosch'1997]

The model $E_{\mathrm{MF}}(\Psi,\Psi^*)$ (with $V(\rho) = g\delta^2(\rho)$) does not survive quantization, but remains a good approximation in some cases (spoiler : too good)!

To « survive quantization » means to stay valid for the same quantum Lagrangian.

no survival → quantum anomaly (smoking gun = deviation from 2ω)

Quantum models featuring PR symmetry : 3D unitary (non-Efimovian) gases, 1D Tonks gas ($V_{1\mathrm{D}}(x) = \infty\,\delta(x)$), $1/r^2$-models in any dimension

PRL 105, 095302 (2010) PHYSICAL REVIEW LETTERS week ending 27 AUGUST 2010

Example of a Quantum Anomaly in the Physics of Ultracold Gases (2D bosons)

Maxim Olshanii,[1,2] Hélène Perrin,[2] and Vincent Lorent[2]

PRL 108, 185303 (2012) PHYSICAL REVIEW LETTERS week ending 4 MAY 2012

Quantum Anomaly, Universal Relations, and Breathing Mode of a Two-Dimensional Fermi Gas (2D BCS-BEC crossover)

Johannes Hofmann*

QM Hamiltonian with the Bethe-Peierls bounday condition $\psi(\rho) \underset{\rho\to 0}{\to} C \ln\left(\frac{\rho}{a_{2D}}\right)$

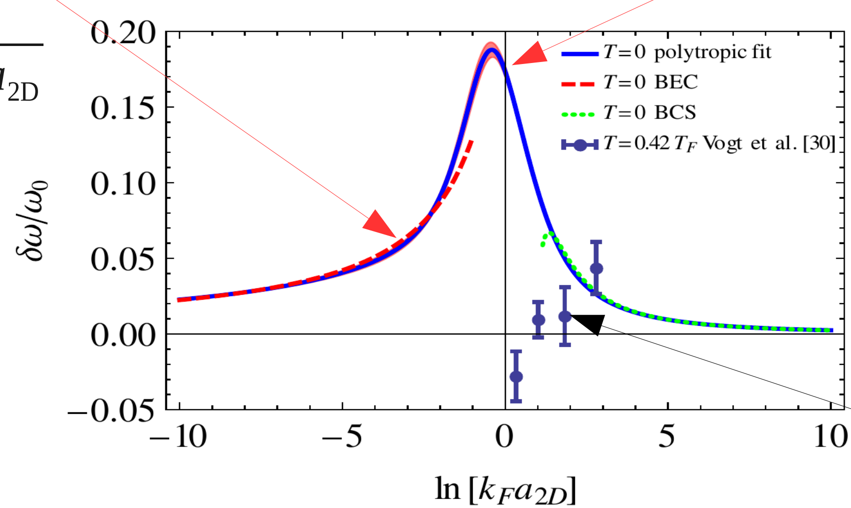
⬇

Hydrodynamic equations with local-density equation of state

Perturbative for bosons [Schick'1971,Popov'1972,Mora&Castin'2009]

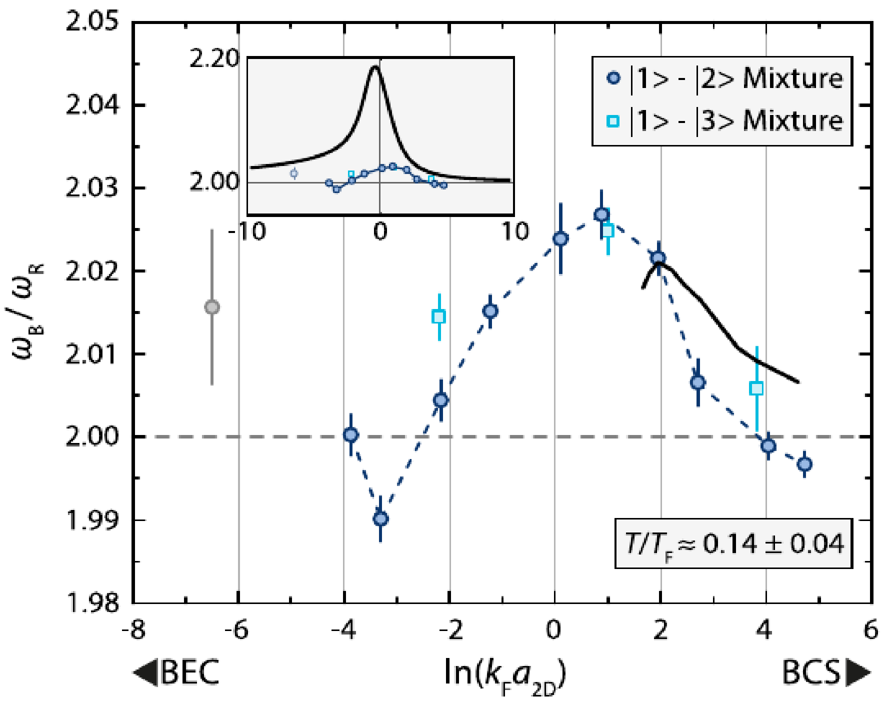
Monte-Carlo in the 2D BCS-BEC crossover [Bertaina&Giorgini'2011]

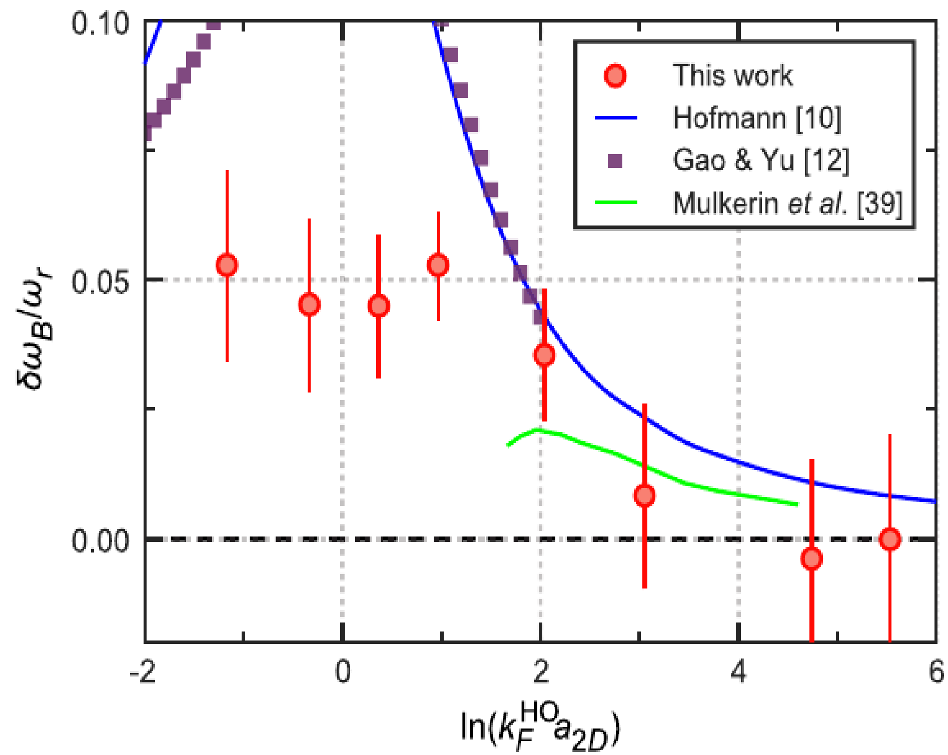
$$\frac{\delta\omega}{\omega} \approx -\frac{1}{4 \ln k_F a_{2D}}$$



[Vogt et al.'2012] (Cambridge)

FERMIONS



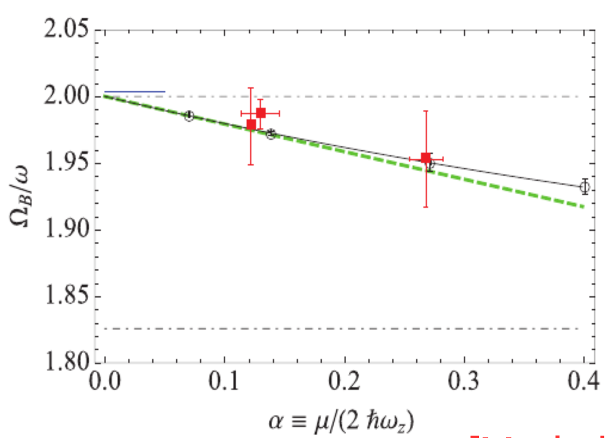


[Holten et al.'2018] (Heidelberg)

see references
for theory works

[Peppler et al.'2018] (Melbourne)

BOSONS



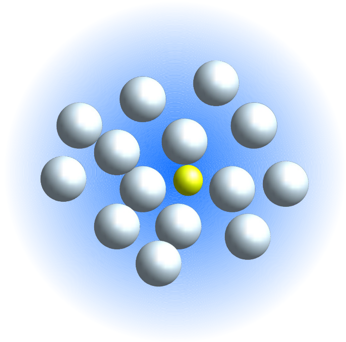
[Merloti et al.'2013] (Paris 13)

Problems :

- small relative shift
- finite-T
- third direction (2D-3D crossover)
- trap anosotropy and inharmonicity

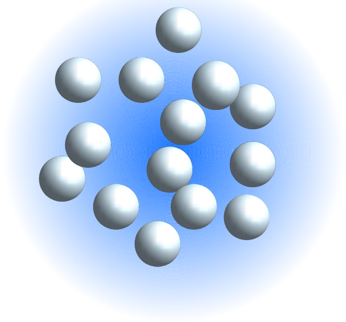
Self-bound

2D N+1 fermionic clusters



Self-bound

2D bosons



If no trap

MF predicts $2\omega=0$ for the breathing mode frequency → ~~small relative shift~~

Self-evaporation, « self-cleaning » → ~~finite-T~~

Relevant (less relevant for smaller *N*) → third direction (2D-3D crossover)

No trap ! → ~~trap anosotropy and inharmonicity~~

Problems :

- ~~small relative shift~~
- ~~finite-T~~
- third direction (2D-3D crossover)
- ~~trap anosotropy and inharmonicity~~

Townes soliton

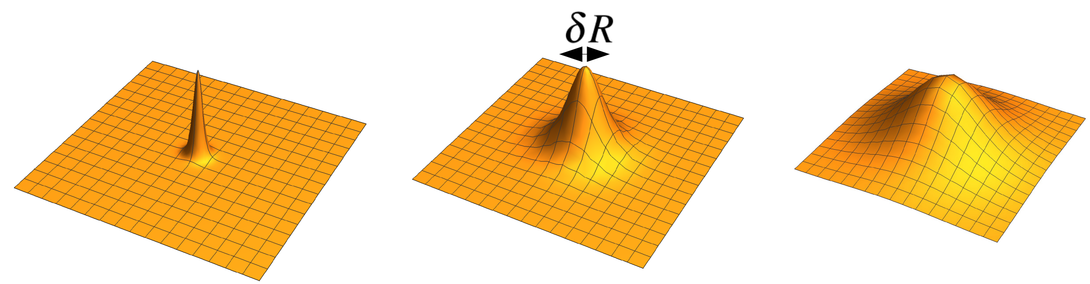
[tutorial: Bakkali-Hassani & Dalibard, Varenna'2022]

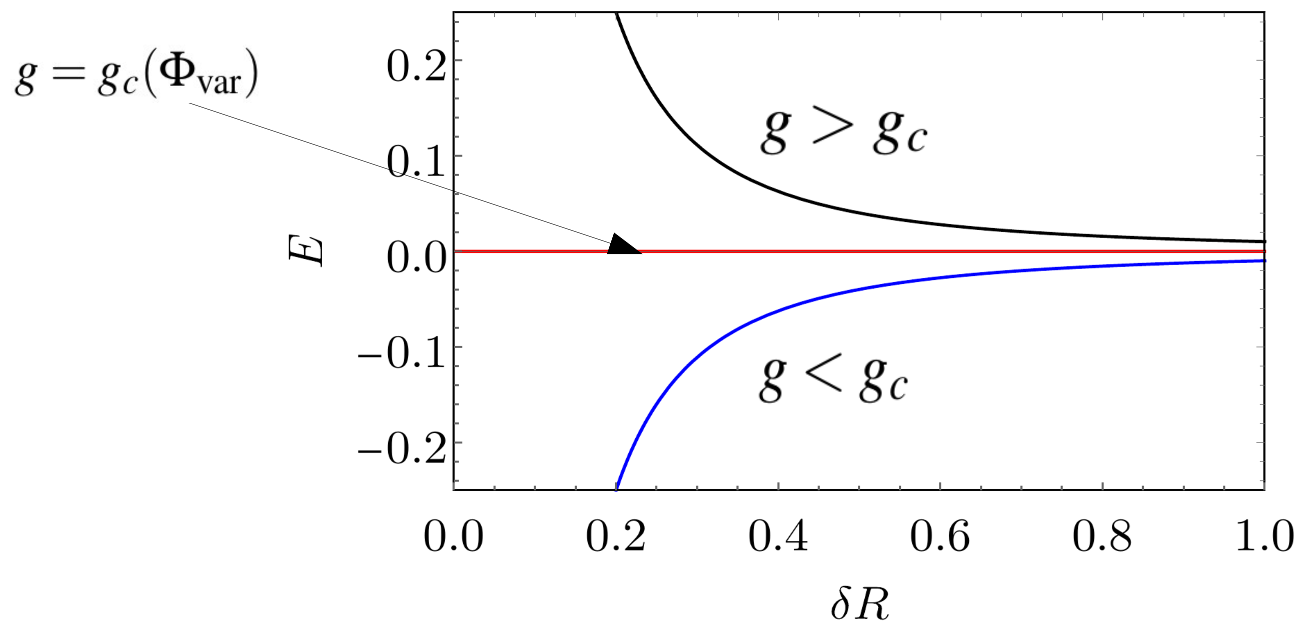
MF scaling invariance

Choose arbitrary profile $\Psi(\rho) = \Phi_{\text{var}}(\rho)$ with $N = \int |\Phi_{\text{var}}(\rho)|^2 d^2\rho$

$E_{\text{MF}}(\Psi,\Psi^*) = (1/2)\int d^2\rho [|\nabla_\rho \Psi(\rho,t)|^2 + g|\Psi(\rho,t)|^4]$ ⇒ $E_0 = E_{\text{MF}}[\Phi_{\text{var}}(\rho)]$

Norm-conserving rescaling $\Phi_{\text{var}}(\rho) \to \Phi_{\text{var}}(\rho/\delta R)/\delta R$ ⇒ $E_0 \to E_0/\delta R^2$





Note that we variationally force the shape to be conserved !

MF scaling invariance

Gross-Pitaevskii equation : $i\partial_t \Psi(\rho,t) = -(1/2)\nabla_\rho^2 \Psi(\rho,t) + g|\Psi(\rho,t)|^2 \Psi(\rho,t)$

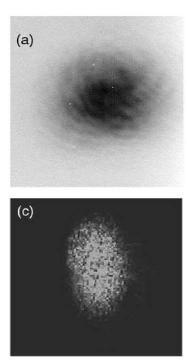
Initial condition : $\Psi(\rho, t=0) = \Phi_{\text{var}}(\rho)$

$$\Psi(\rho,t)$$

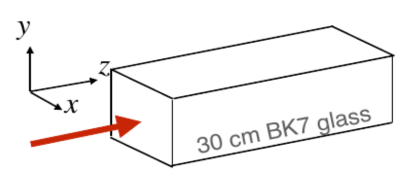
A few facts about this dynamics : $N \frac{d^2}{dt^2}\langle \rho^2 \rangle = 4E$ [Vlasov, Petrishchev,Talanov'71, Pitaevskii'96]

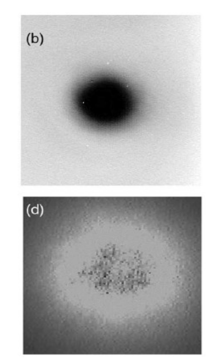
$$\frac{1}{N}\int |\Psi(\rho,t)|^2 \rho^2 d^2\rho \qquad E_{\text{MF}}[\Psi(\rho,0), \Psi^*(\rho,0)]$$

Optical wave collapse
[Moll, Gaeta, and Fibich'2003]



Their z is our t





Input : randomly distorted or elliptic ⇒ Output : clean isotropic ``Townsian"

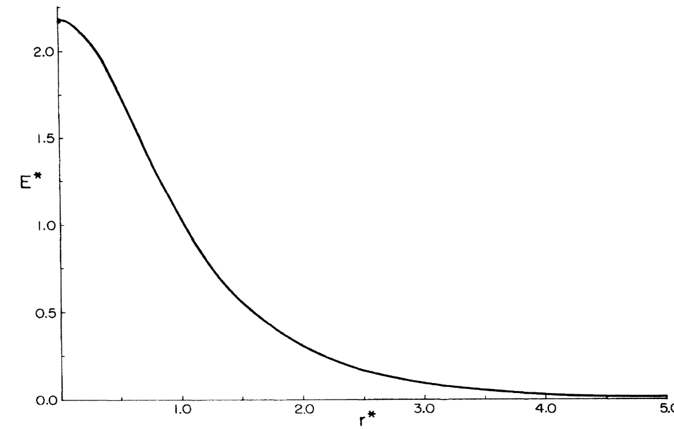
Townes soliton [Chiao, Garmire, Townes'1964]

Townes profile [Chiao, Garmire, Townes'1964] ⟹ $$\frac{d^2E^*(r^*)}{dr^{*2}} + \frac{1}{r^*}\frac{dE^*(r^*)}{dr^*} - E^*(r^*) + E^{*3}(r^*) = 0$$

$$-f''(r) - f'(r)/r - f(r)^3 = -f(r)$$

$$C := \int_0^\infty drrf^2(r) = \int_0^\infty drr[f'(r)]^2 = \frac{1}{2}\int_0^\infty drrf^4(r) = 1.862$$

$$M_2 := \int_0^\infty drr^3 f^2(r) = 2.211$$



⇓

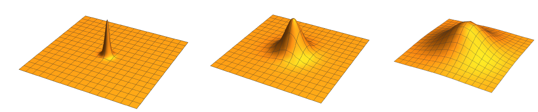
The stationary state (*R* is arbitrary but time-independent)

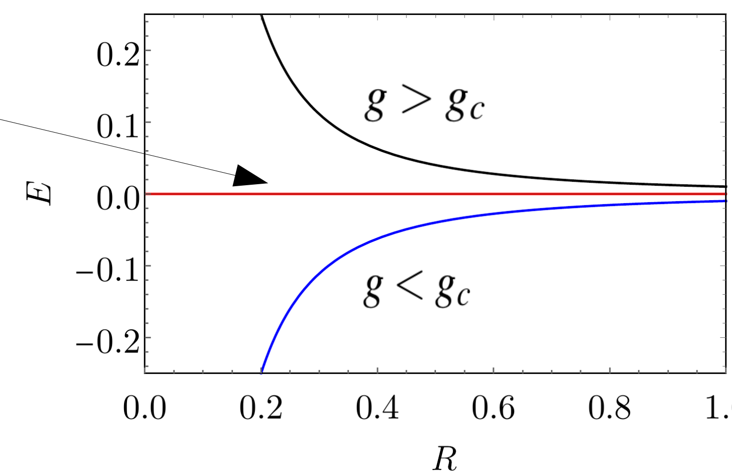
$$\Psi_R(\rho,t) = e^{it/(2R^2)}\Psi_R(\rho) = e^{it/(2R^2)}\sqrt{N/(2\pi C)}f(\rho/R)/R$$

solves GPE

$$i\partial_t\Psi(\rho,t) = -(1/2)\nabla_\rho^2\Psi(\rho,t) + g|\Psi(\rho,t)|^2\Psi(\rho,t)$$

Under the condition $g = g_c = -\pi C/N = -5.85/N$





Dynamics of Townes soliton

Consider the anzats $\Psi(\rho,t) = \sqrt{N/(2\pi C)}e^{i\theta(\rho,t)} f[\rho/R(t)]/R(t)$

Determine $\theta(\rho,t)$ and $R(t)$ by minimizing the action $S = \int dt d^2\rho \mathscr{L}(\Psi,\Psi^*)$ with Lagrangian density

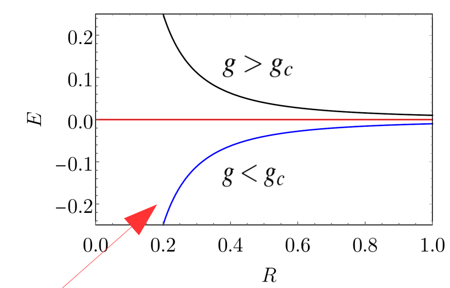
$$\mathscr{L}(\Psi,\Psi^*) = \mathrm{Re}[i\Psi^*(\rho,t)\partial_t\Psi(\rho,t)] - |\nabla_\rho\Psi(\rho,t)|^2/2 - g|\Psi(\rho,t)|^4/2$$

Minimization wrt $\theta(\rho,t)$

Continuity equation : $\partial_t|\Psi|^2 + \nabla_\rho(|\Psi|^2\nabla_\rho\theta) = 0$

$$\theta(\rho,t) = [\dot{R}(t)/R(t)]\rho^2/2 + \phi(t)$$

$$S = \mathrm{const} + \int dt\, \{NM_2\dot{R}^2(t)/(2C) - (g-g_c)N^2/[2\pi CR^2(t)]\}$$



Classical motion of a particle of mass $m_{\mathrm{eff}} = NM_2/C$ in external potential $(g-g_c)N^2/(2\pi CR^2)$

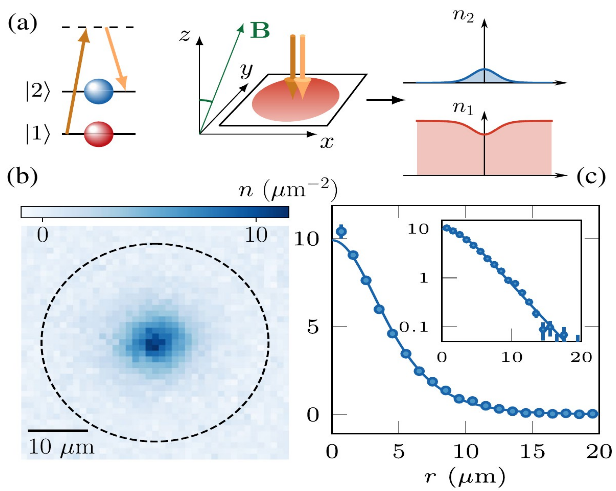
with total energy $E = m_{\mathrm{eff}}\dot{R}^2/2 + (g-g_c)N^2/(2\pi CR^2) = \mathrm{const}$ ...check that $\partial_t^2\sigma^2 = 4E_{\mathrm{MF}}/N$

$$R(t) = \sqrt{(g-g_c)N/(\pi A) + A(t-t_0)^2/M_2}$$

where A and t_0 are determined by $R(0)$ and $\dot{R}(0)$

Collège de France Rb exp

[Bakkali-Hassani et al.'21, see also Bakkali-Hassani & Dalibard, Varenna'2022]



3D scattering lengths
a_{11}=100.9 a_0
a_{12}=100.4 a_0
a_{22}= 94.9 a_0

2D coupling strengths
g_{11}=0.160
g_{12}=0.159
g_{22}=0.151

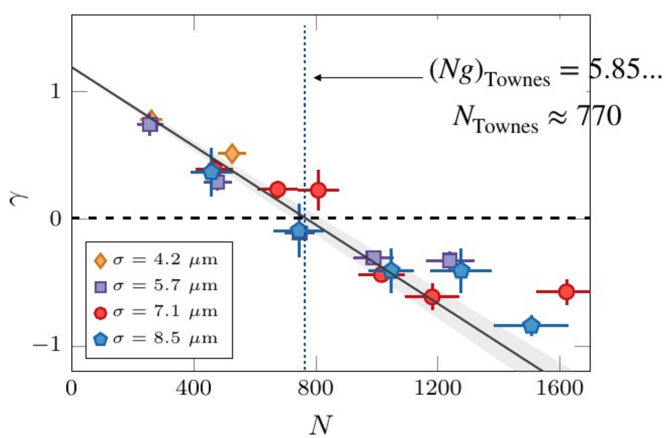
$$g_{\rm eff} = g_{22} - \frac{g_{12}^2}{g_{11}} \approx -0.0076$$

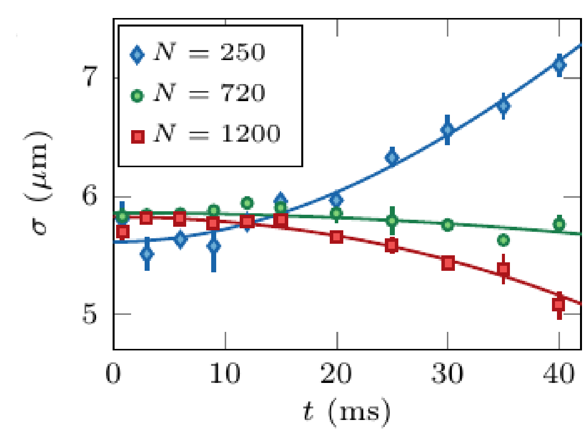
$$N_{\rm Townes}\,|g| = 5.85$$

$$N_{\rm Townes} \approx 770$$

Expansion factor

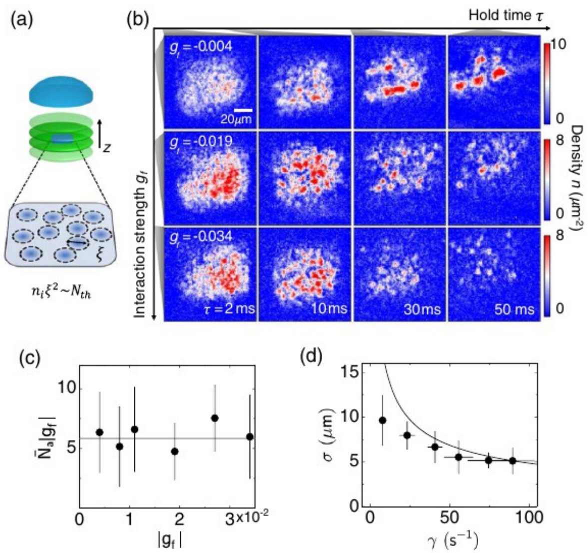
$$\propto \frac{\mathrm{d}}{\mathrm{d}t}\langle r^2 \rangle$$

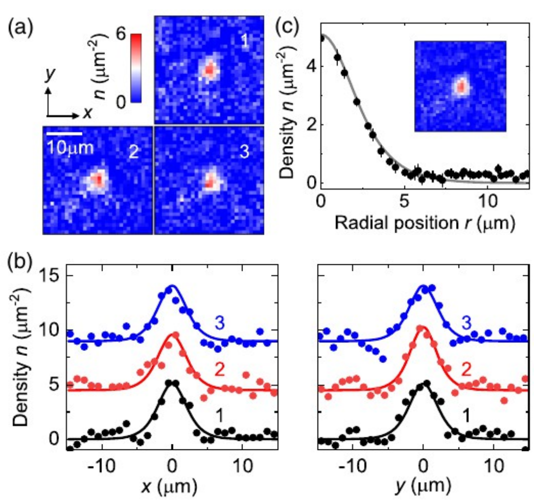




Purdue Cs exp

[Chen&Hung'20]





prepare repulsive
2D Cs condensate

$$g = \sqrt{8\pi}a_{3D}/l_\perp > 0$$

quench to

$$g < 0$$

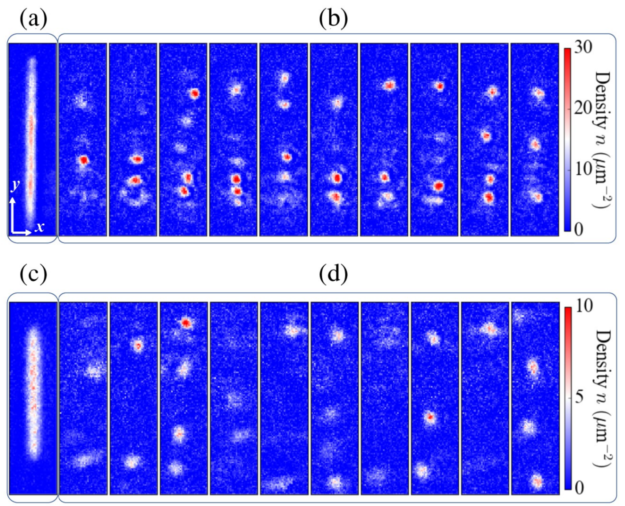
Depending on g and init density → Modulational instability gives solitons with more or less expected R and N

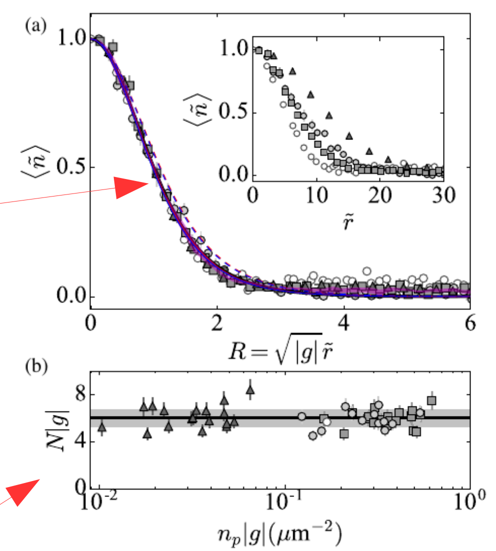
« self-cleaned »
|Townes profile|^2

$$|\Psi_R(\rho)|^2 = \frac{N}{2\pi C R^2} f^2(\rho/R)$$

$$g = g_c = -\pi C/N = -5.85/N$$

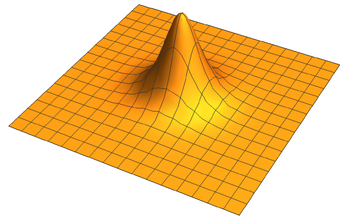
[Chen&Hung'21]



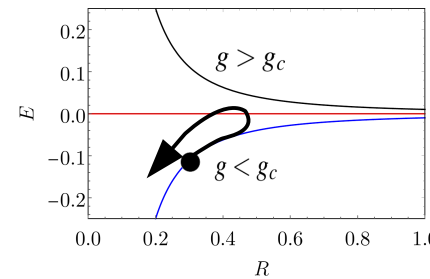


Summary (up to this point)

Townes soliton :



$$g = g_c = -5.85/N$$



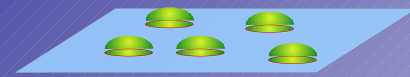
- universal profile
- Self-similar dynamics parametrized by collective variable $R(t)$
- No periodic dynamics, no collective excitations (self-evaporation or self-cleaning)
- Vanishing breating mode frequency
- If stationary, $E = Kin + Int = 0$ ⇨ automatic cancellation of the MF energy

⇩

- higher-order terms become important ! Beyond-mean-field terms may be too weak in current expts. Need lower *N* ...
- $1/R^2$ scaling of the energy... relation to Efimov ?

Few-body results

Few-body studies



3 bosons Bruch&Tjon'1979: no Thomas collapse, no Efimov effect, two trimer states ← Good for lifetime !

$$B_3 = 16.522688(1)\,B_2 \qquad B_3^{ex} = 1.2704091(1)\,B_2$$

[Bruch&Tjon'79; Adhikari et al.'88; Nielsen,Fedorov&Jensen'99; Hammer&Son'04; Kartavtsev&Malykh'06...]

4 bosons : $B_4 = 197.3(1)\,B_2 \qquad B_4^{ex} = 25.5(1)\,B_2$

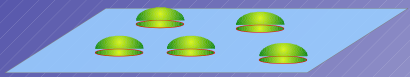
[Platter,Hammer&Meissner'04; Brodsky et al'06]

$N \to \infty$ Hammer&Son'04;

$$B_N / B_{N-1} \underset{N\to\infty}{\to} e^{4/C} = 8.567\ldots \quad \Leftrightarrow \quad B_N \propto B_2 e^{4N/C}$$

$$R_N / R_{N-1} \underset{N\to\infty}{\to} e^{-2/C} = 0.3417\ldots \quad \Leftrightarrow \quad R_N \propto a_{2D} e^{-2N/C}$$

Few-body studies



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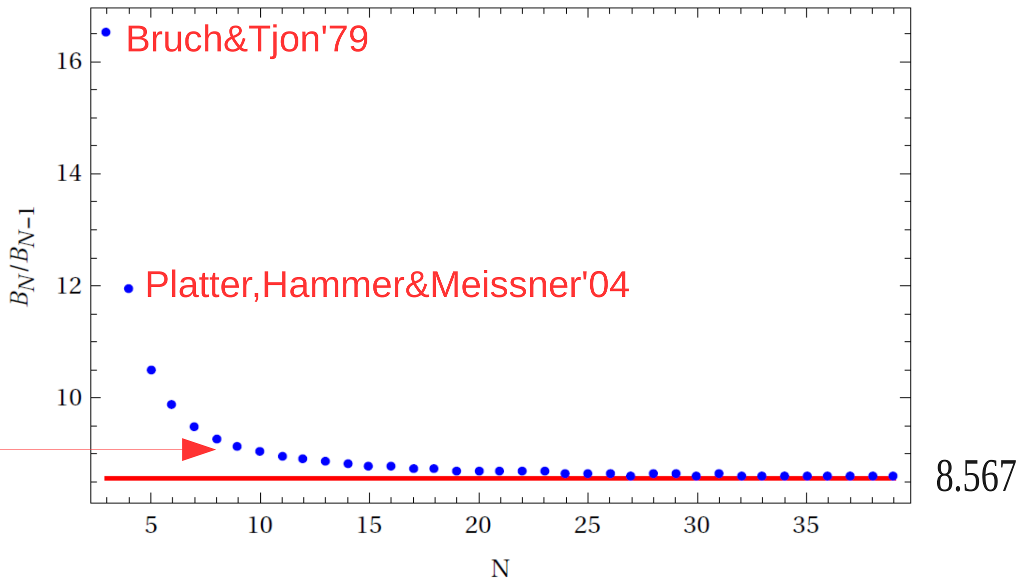
[Platter,Hammer&Meissner'04; Brodsky et al'06]

$N \leq 7$ finite-range calculations
[Blume'05] (inconclusive)

$N \leq 10$ lattice EFT
[Lee'06] $B_N / B_{N-1} \to 8.3(6)$

$N \leq 26$ STM-DMC approach
[Bazak&DSP'18]



For N>4 no information about excited states:(

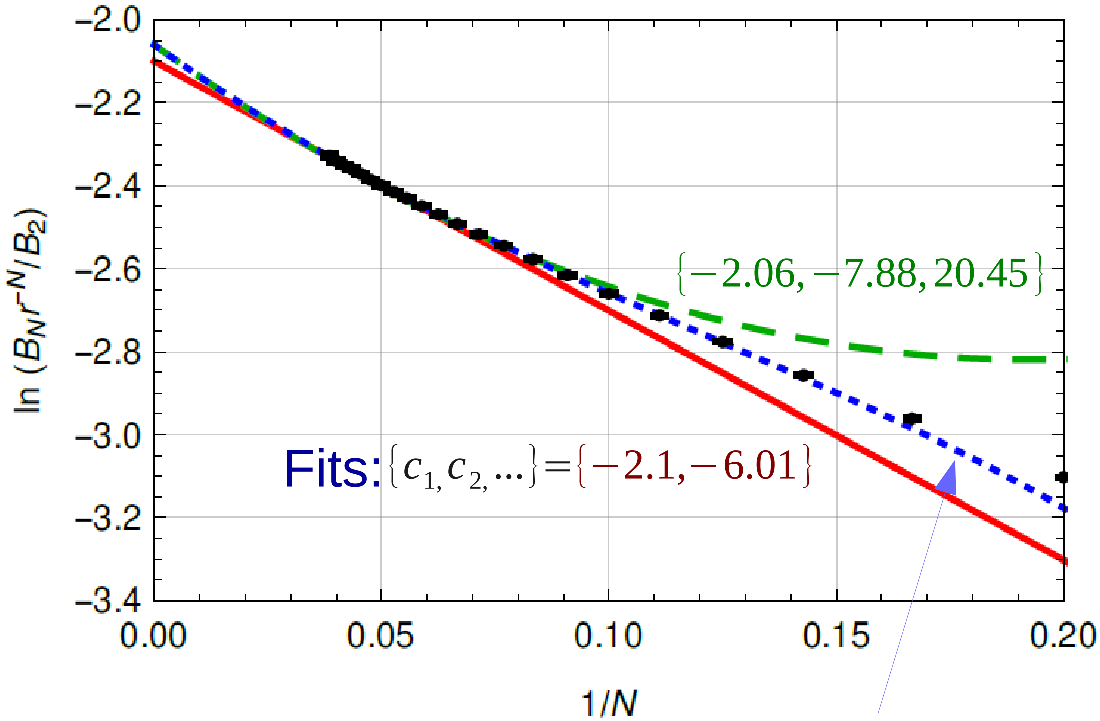
STM-DMC [Bazak&DSP'18]

N	B_N/B_2	N	B_N/B_2
3	$1.65225(2)\times10^{1}$	15	$8.135(2)\times10^{12}$
4	$1.9720(1)\times10^{2}$	16	$7.129(4)\times10^{13}$
5	$2.0745(1)\times10^{3}$	17	$6.232(2)\times10^{14}$
6	$2.0471(1)\times10^{4}$	18	$5.438(3)\times10^{15}$
7	$1.9462(1)\times10^{5}$	19	$4.734(2)\times10^{16}$
8	$1.8070(1)\times10^{6}$	20	$4.119(2)\times10^{17}$
9	$1.6508(4)\times10^{7}$	21	$3.577(2)\times10^{18}$
10	$1.4905(2)\times10^{8}$	22	$3.108(4)\times10^{19}$
11	$1.3345(2)\times10^{9}$	23	$2.694(5)\times10^{20}$
12	$1.1873(4)\times10^{10}$	24	$2.332(4)\times10^{21}$
13	$1.0508(3)\times10^{11}$	25	$2.018(4)\times10^{22}$
14	$9.2596(9)\times10^{11}$	26	$1.748(4)\times10^{23}$

Conjecture...wrong !

$$B_N = B_2\, e^{\ln(8.567)N + c_1 + c_2/N + \ldots}$$

$$\ln\left(B_N\, 8.567^{-N}/B_2\right) = c_1 + c_2/N + \ldots$$



$\{-2.06, -7.94, 27.2, -77\}$

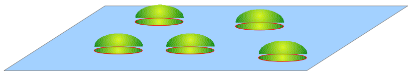
Updated conjecture [this work] :

$$B_N = B_2\, e^{4N/C + c_1 - 2\sqrt{2}/\sqrt{M_2 N} + \ldots}$$

MF LO BMF « free bonus »

Bogoliubov analysis

Renormalization



$$\hat{H} = \frac{1}{2}\int d^2\rho\left(-\hat{\Psi}^\dagger_\rho \nabla^2_\rho \hat{\Psi}_\rho + g\hat{\Psi}^\dagger_\rho \hat{\Psi}^\dagger_\rho \hat{\Psi}_\rho \hat{\Psi}_\rho\right)$$ - needs regularization

square lattice

$$\hat{H} = \sum_{k_x,k_y\in[-\pi/h,\pi/h]} \varepsilon^{(0)}_{\mathbf{k}} \hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}} + \frac{g}{2}\sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}} \hat{a}^\dagger_{\mathbf{k}_1+\mathbf{q}} \hat{a}^\dagger_{\mathbf{k}_2-\mathbf{q}} \hat{a}_{\mathbf{k}_1} \hat{a}_{\mathbf{k}_2}$$

$$\varepsilon^{(0)}_{\mathbf{k}} = [2-\cos(k_x h) - \cos(k_y h)]/h^2 \approx k^2/2$$

cutoff

$$\hat{H} = \sum_{\mathbf{k}} \frac{k^2}{2} \hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}} + \frac{1}{2}\sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}} \hat{a}^\dagger_{\mathbf{k}_1+\mathbf{q}} \hat{a}^\dagger_{\mathbf{k}_2-\mathbf{q}} U(\mathbf{q}) \hat{a}_{\mathbf{k}_1} \hat{a}_{\mathbf{k}_2}$$

$$U(\mathbf{q}) = \begin{cases} g, \text{ for } |\mathbf{q}| < \kappa \\ 0, \text{ for } |\mathbf{q}| > \kappa \end{cases}$$

Both models should correspond to $\psi(\rho) \underset{\rho\to 0}{\to} C\ln\left(\frac{\rho}{a_{2D}}\right)$ or, equivalently, reproduce $B_2 = -4e^{-2\gamma}/a^2_{2D}$

$$\frac{1}{g} \approx \frac{\ln(|B_2|h^2/32)}{4\pi}$$

$$\frac{1}{g} \approx \frac{\ln(|B_2|/\kappa^2)}{4\pi}$$

Both valid when $|g| \ll 1$ i.e., for large (repulsion) or small (attraction) $|B_2|/\kappa^2$

Trade a single (universal) interaction parameter B_2<0 for a pair (g,κ), but can now do perturbation theory in $|g|$<<1!

Bogoliubov for homogeneous gas

$$\hat{H} = \sum_{\mathbf{k}} \frac{k^2}{2} \hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}} + \frac{1}{2} \sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}} \hat{a}^\dagger_{\mathbf{k}_1+\mathbf{q}} \hat{a}^\dagger_{\mathbf{k}_2-\mathbf{q}} U(\mathbf{q}) \hat{a}_{\mathbf{k}_1} \hat{a}_{\mathbf{k}_2}$$

$\hat{a}_0, \hat{a}^\dagger_0 \to a_0$ - assume real

$$\hat{H} = H_0 + \hat{H}_2 + \hat{H}_3 + \hat{H}_4$$

$$H_0 = \frac{1}{2} g a_0^4 \qquad a_0^2 = n - \sum_{\mathbf{k}}{}' \hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}}$$

$$\hat{H}_2 = \sum_{\mathbf{k}}{}' \left[\frac{k^2}{2} + g a_0^2 \right] \hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}} + \frac{1}{2} \sum_{\mathbf{k}}{}' U(\mathbf{k}) a_0^2 (\hat{a}^\dagger_{\boldsymbol{k}} \hat{a}^\dagger_{-\boldsymbol{k}} + \hat{a}_{\boldsymbol{k}} \hat{a}_{-\boldsymbol{k}} + 2 \hat{a}^\dagger_{\boldsymbol{k}} \hat{a}_{\boldsymbol{k}})$$

$$\hat{H}_3 = \sum_{\boldsymbol{k}_1,\boldsymbol{k}_2}{}' U(\mathbf{k}_1) a_0 \hat{a}^\dagger_{\boldsymbol{k}_1+\boldsymbol{k}_2} (\hat{a}_{\boldsymbol{k}_1} + \hat{a}^\dagger_{-\boldsymbol{k}_1}) \hat{a}_{\boldsymbol{k}_2}$$

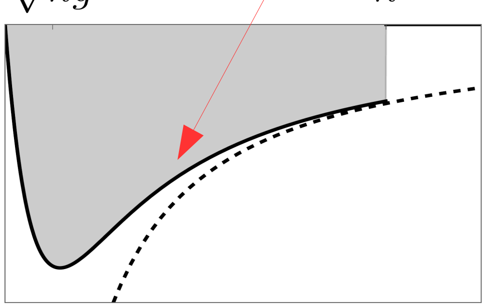
$$\hat{H}_4 = \frac{1}{2} \sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}}{}' \hat{a}^\dagger_{\mathbf{k}_1+\mathbf{q}} \hat{a}^\dagger_{\mathbf{k}_2-\mathbf{q}} U(\mathbf{q}) \hat{a}_{\mathbf{k}_1} \hat{a}_{\mathbf{k}_2}$$

Diagonalization

$$\frac{E}{\text{Surface}} = \frac{g n^2}{2} + \frac{1}{2} \sum_{k}^{\kappa} \left[\sqrt{k^4/4 + gnk^2} - k^2/2 - gn \right] \approx \frac{g n^2}{2} + \frac{g^2 n^2}{8\pi} \ln \frac{gn\sqrt{e}}{\kappa^2}$$

Higher orders
[Mora&Castin'2009]

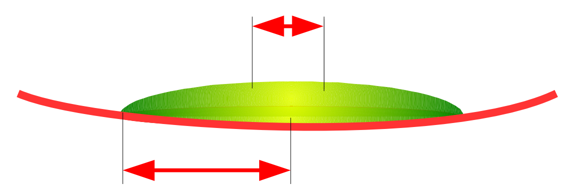
$$+ \sim n^2 g^3$$



$$\propto \int \frac{dk}{k}$$

BMF correction is local !

Healing length $\xi = 1/\sqrt{gn}$



$R_{\mathrm{TF}} = \sqrt{2gn}/\omega$

Local-density approx when $\omega \ll gn$

Inhomogeneous Bogoliubov [this work]

$$\hat{H} = \frac{1}{2}\int d^2\rho(-\hat{\Psi}^\dagger_\rho \nabla^2_\rho \hat{\Psi}_\rho + g\hat{\Psi}^\dagger_\rho \hat{\Psi}^\dagger_\rho \hat{\Psi}_\rho \hat{\Psi}_\rho)$$

Cannot do LDA !!!

$$1/\sqrt{|g|n} \sim R$$

$$\hat{\Psi}_\rho = \Psi_R(\rho) + \delta\hat{\Psi}_\rho$$

$$E_{\mathrm{MF}} = (g - g_c)N^2/(2\pi C R^2) \quad + \quad \hat{H}_2 = \frac{1}{2}\int d^2\rho \begin{pmatrix}\delta\hat{\Psi}^\dagger_\rho & \delta\hat{\Psi}_\rho\end{pmatrix} \begin{pmatrix}\hat{A} & \hat{B} \\ \hat{B} & \hat{A}\end{pmatrix} \begin{pmatrix}\delta\hat{\Psi}_\rho \\ \delta\hat{\Psi}^\dagger_\rho\end{pmatrix} - \mathrm{Tr}(\hat{A})/2$$

$$\hat{A} = -\nabla^2_\rho/2 - \mu + 2g_c\Psi^2_R(\rho),\ \hat{B} = g_c\Psi^2_R(\rho)$$

Almost complete cancellation Kin + Int

$$|g - g_c| \sim 1/N^2 \ll |g| \approx \pi C/N$$

Blaizot & Ripka
"Quantum Theory of Finite Systems"

Inhomogeneous Bogoliubov [this work]

$$\hat{H} = \frac{1}{2}\int d^2\rho\,(-\hat{\Psi}^\dagger_\rho \nabla^2_\rho \hat{\Psi}_\rho + g\hat{\Psi}^\dagger_\rho\hat{\Psi}^\dagger_\rho\hat{\Psi}_\rho\hat{\Psi}_\rho)$$

$$\hat{\Psi}_\rho = \Psi_R(\rho) + \delta\hat{\Psi}_\rho$$

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$$1/\sqrt{|g|n} \sim R$$

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Almost complete cancellation Kin + Int

$$|g - g_c| \sim 1/N^2 \ll |g| \approx \pi C/N$$

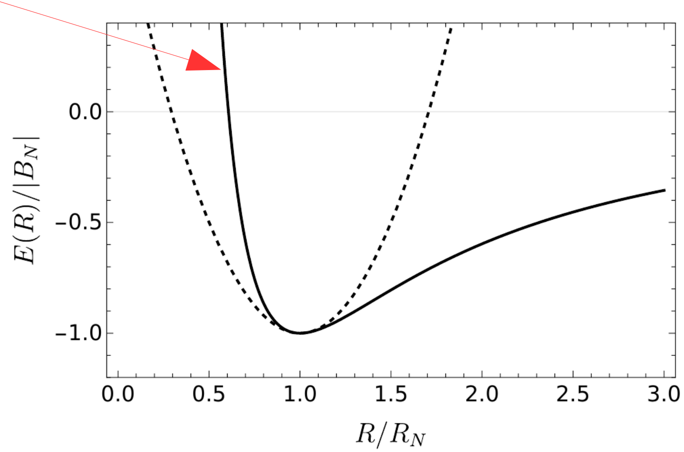
Blaizot & Ripka
"Quantum Theory of Finite Systems"

$$E(R) = -2|B_N|\frac{R_N^2}{R^2}\ln\frac{R\sqrt{e}}{R_N}$$

$$R_N = \sqrt{\frac{C}{8|B_2|}}e^{-2N/C - c_1/2}$$

$$B_N = B_2 e^{4N/C + c_1}$$

$$c_1 = -1.91(1)$$

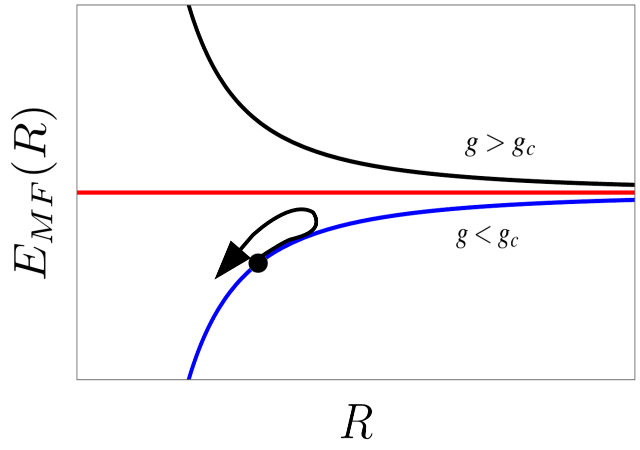


Breathing dynamics [this work]

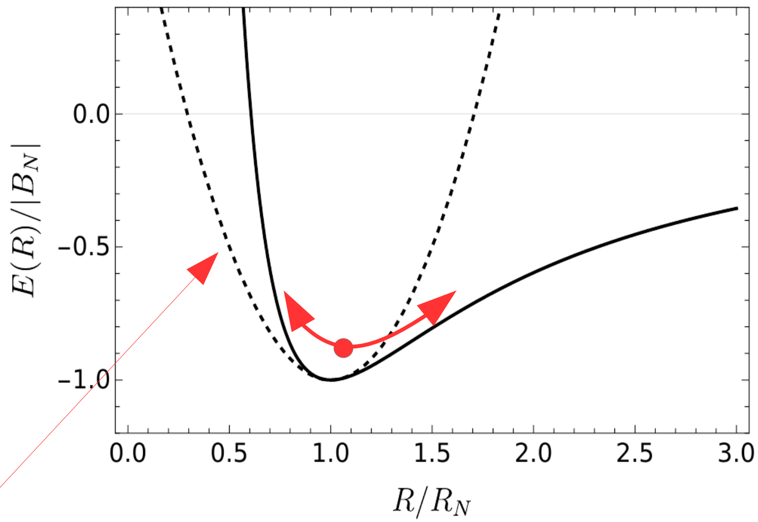
$$E(R) = -2|B_N|\frac{R_N^2}{R^2}\ln\frac{R\sqrt{e}}{R_N}$$

Classical dynamics of collective coordinate *R*

$$S = \int dt\left[\frac{NM_2}{C}\frac{\dot{R}^2(t)}{2} - E[R(t)]\right]$$



+ BMF corr.



Harmonic approx.

low-amplitude breathing mode frequency :

$$\Omega = \frac{4\sqrt{2}}{\sqrt{NM_2}}|B_N|$$

$$\tau = 2\pi/\Omega = 7.1mR_N^2\sqrt{N}/\hbar \quad \Longrightarrow \quad \tau \approx 100\text{ms}$$

Cs mass

$$N = 16$$

$$R_N = 1.3\mu\text{m}$$

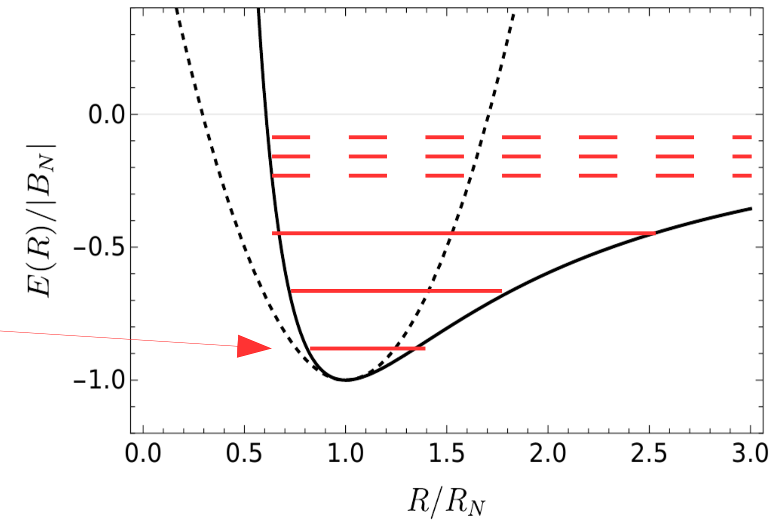
Conjectures!

quantize $S = \int dt \left[\frac{NM_2}{C} \frac{\dot{R}^2(t)}{2} - E[R(t)] \right]$

$$\Omega = \frac{4\sqrt{2}}{\sqrt{NM_2}} |B_N|$$

Discrete « breathing » spectrum

With level spacing/well depth ~ $N^{-1/2}$



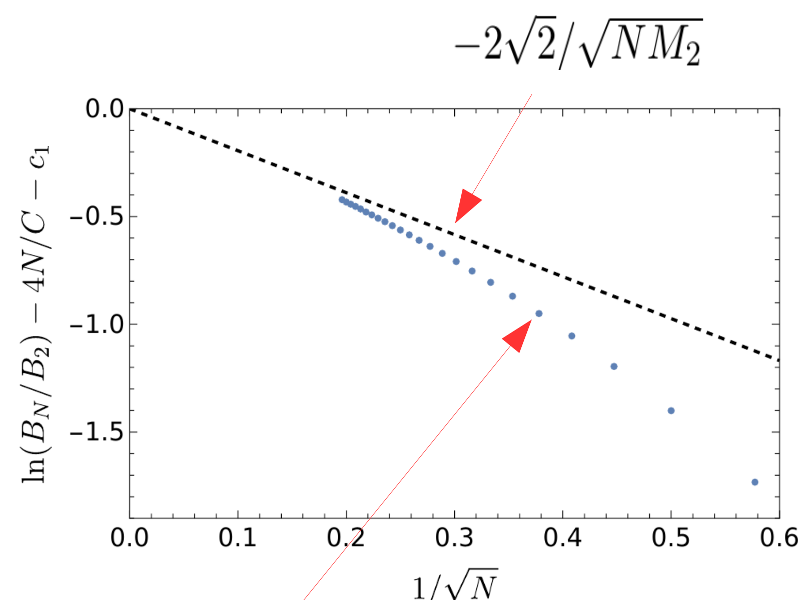
$$B_3^{ex} = 1.2704091(1) B_2 \qquad B_4^{ex} = 25.5(1) B_2$$
$$B_3 = 16.522688(1) B_2 \qquad B_4 = 197.3(1) B_2$$

Conjecture #1 : with increasing N there will be second, third,... excited states

Conjecture #2 : include zero-point energy $\Omega/2$

$$B_N \to B_N + \Omega/2 \approx B_2 e^{4N/C + c_1 - 2\sqrt{2}/\sqrt{M_2 N}}$$

Very special term due to quantum anomaly. Otherwise. expect integer powers of $g \sim 1/N$ (cf. repulsive case [Mora&Castin'2009])



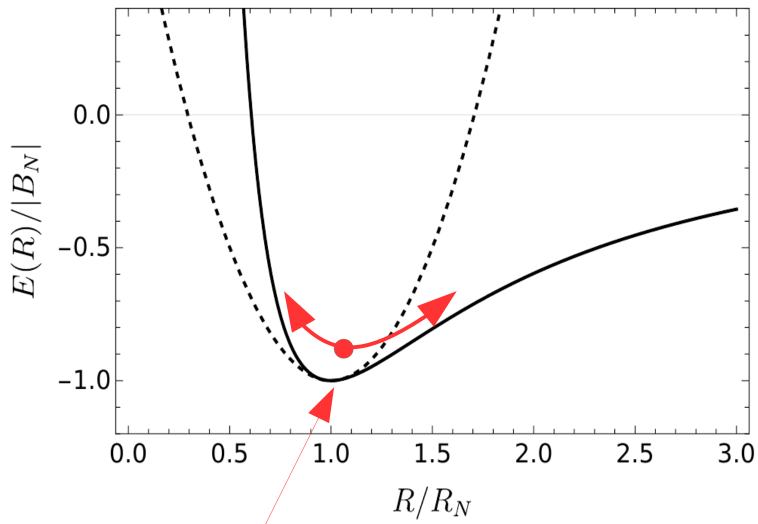
[Bazak&DSP'18]

Summary/Outlook

$$B_N \approx B_2 e^{4N/C + c_1 - 2\sqrt{2}/\sqrt{M_2 N}}$$

Result of the Bogoliubov theory

nonanalytic power of 1/*N* due to quantum anomaly (is this general?)



Observable quantum effect - breathing mode with frequency

$$\Omega = \frac{4\sqrt{2}}{\sqrt{NM_2}}|B_N|$$

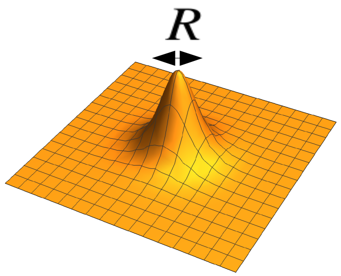
To do :

- excited states for moderate *N*
- precise ground-state energy for *N*~100
- Quantization of the breathing mode : nonequidistant spectrum : dynamics ?
- Nonlinear excitations of Townes solitons
- Account of the third dimension : 2D → quasi-2D ?

cf. [Suchorowski et al. 2025]

Thank you !

Hammer&Son'2004: $g \to g(R) = \frac{4\pi}{\ln(|B_2|R^2 \times \mathrm{const})}$



$\Psi_R(\rho) = \sqrt{N/(2\pi C)} f(\rho/R)/R$

Renormalized coupling constant

$$E_{\mathrm{MF}}(R) = (1/2) \int d^2\rho [|\nabla_\rho \Psi_R(\rho)|^2 + g(R)|\Psi_R(\rho)|^4]$$

$E_{\mathrm{MF}}(R) = [g(R) - g_c] N^2/(2\pi C R^2)$ has minimim at $R_{\min}$: $g(R_{\min}) = g_c + O(N^{-2})$

$g_c = -\pi C/N = -5.85/N$

$$R_{min} = R_N \propto |B_2|^{-1/2} e^{-2N/C}$$

$$B_N \propto B_2 e^{4N/C}$$

Independence of the cutoff

$$\frac{E}{\text{Surface}} = \frac{gn^2}{2} + \frac{1}{2}\sum_k^{\kappa} [\sqrt{k^4/4+gnk^2} - k^2/2 - gn] \approx \frac{gn^2}{2} + \frac{g^2 n^2}{8\pi} \ln \frac{gn\sqrt{e}}{\kappa^2}$$

$$g \approx 4\pi/\ln(|B_2|/\kappa^2) \Longrightarrow dg/d\kappa = -g^2/(2\pi\kappa) \Longrightarrow dE/d\kappa \propto g^3$$

Beyond Bogoliubov

Cutoff independence on the Bogoliubov level
(g just has to remain small)

$$\kappa^2 = gn\sqrt{e}$$

Implicit eq. : $g = \frac{4\pi}{\ln \frac{|B_2|}{gn\sqrt{e}}} \Longrightarrow g(n, B_2)$

$$\frac{E}{\text{Surface}} = \frac{g(n,B_2)n^2}{2}$$

cf. [Schick'1971]:

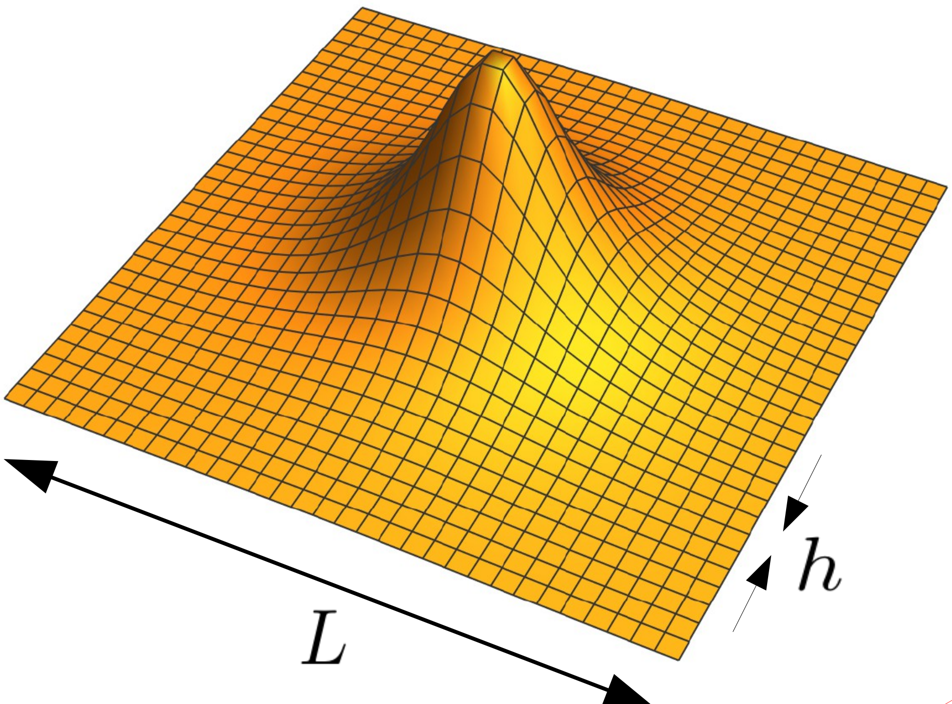
$$g = \frac{4\pi}{\ln \frac{1}{a_{2D}^2 n}}$$

Leading BMF result [Popov'1972]
LHY (or Bogoliubov) accuracy

Jargon : « density-dependent interaction» ⟹ LDA +GPE ...

Calculation of c_1

$$\hat{L}_h \Phi_{i,j} = (\Phi_{i+1,j} + \Phi_{i-1,j} + \Phi_{i,j+1} + \Phi_{i,j-1} - 4\Phi_{i,j})/h^2$$



$$\hat{A} = -\hat{L}_h/2 - \mu + 2g_c \Psi^2_{R=1}(\boldsymbol{\rho}_{i,j})$$

$$\hat{B} = g_c \Psi^2_{R=1}(\boldsymbol{\rho}_{i,j})$$

$$\begin{pmatrix} \hat{A} & \hat{B} \\ -\hat{B} & -\hat{A} \end{pmatrix} \begin{pmatrix} u_\nu(\boldsymbol{\rho}) \\ v_\nu(\boldsymbol{\rho}) \end{pmatrix} = \epsilon_\nu \begin{pmatrix} u_\nu(\boldsymbol{\rho}) \\ v_\nu(\boldsymbol{\rho}) \end{pmatrix}$$

$$E_{\text{BMF}} = \sum_{\nu:\varepsilon_\nu > 0} \varepsilon_\nu/2 - \text{Tr}(\hat{A})/2$$

$$c_1(L,h) = -(8/C)E_{\text{BMF}}(L,h) - 1 - 8\ln 2 + \ln C + 2\ln h$$

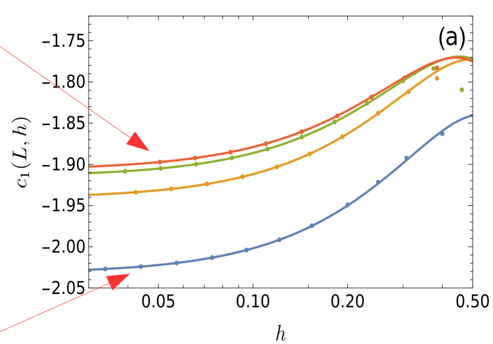
Extrap. to $h \to 0$

$L = 20$

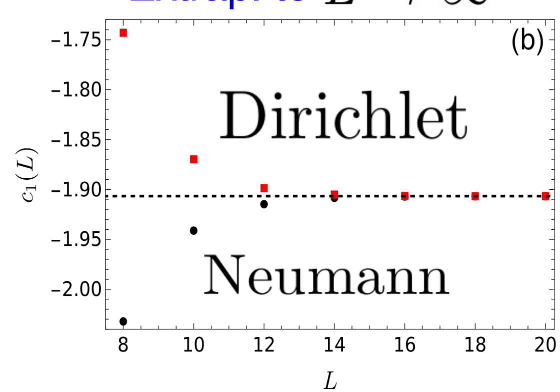
$L = 12$

$L = 10$

$L = 8$



Extrap. to $L \to \infty$



$$c_1 = -1.9067$$