Bose-Einstein condensation on curved surfaces

Natália Salomé Móller
Research Center for Quantum Information
Slovak Academy of Sciences

Workshop and School on
"Frontiers in ultracold quantum gases"
10th June 2026

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Relativistic Quantum Information
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Investigando a natureza da gravidade com os métodos da informação quântica
Revista Brasileira de Ensino de Física **47** (Suppl 3), e20250371 (2025)

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Exploring the nature of gravity with quantum information methods
arXiv:2512.20429 (2025)

BEC on a bubble trap

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A two-dimensional manifold that is embedded in $\mathbb{R}^3$

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Manifold:
A mathematical object to which we can associate notions such as length and curvature.

It can be always embedded in $\mathbb{R}^n$; but it is not needed.

Surface:
A two-dimensional manifold that is embedded in $\mathbb{R}^3$

Matter tells spacetime how to curve, and curved spacetime tells matter how to move.

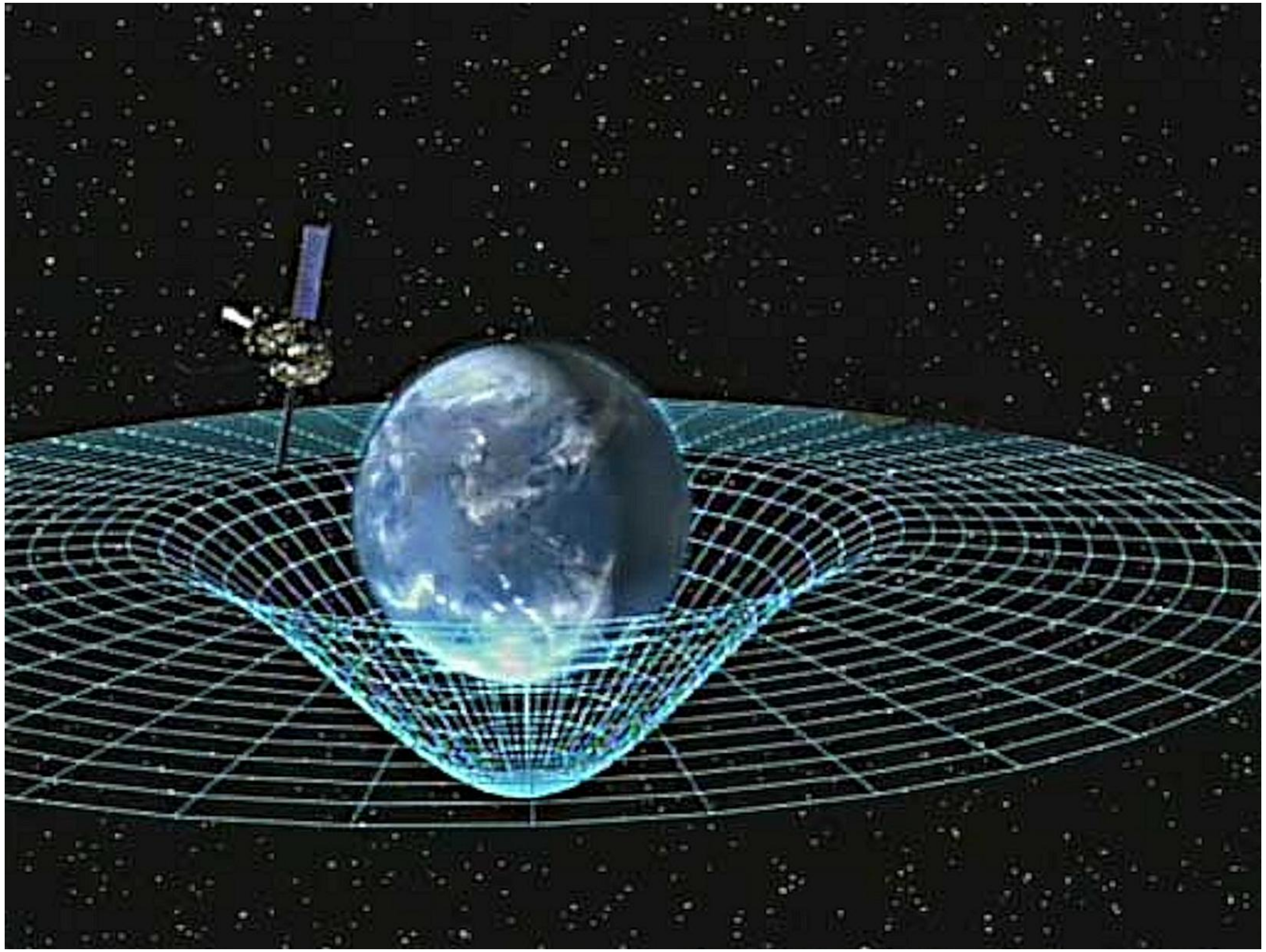

Figure: Wikipedia.

To describe a curved Universe, we need the concept of **manifold**, since there is no outside.

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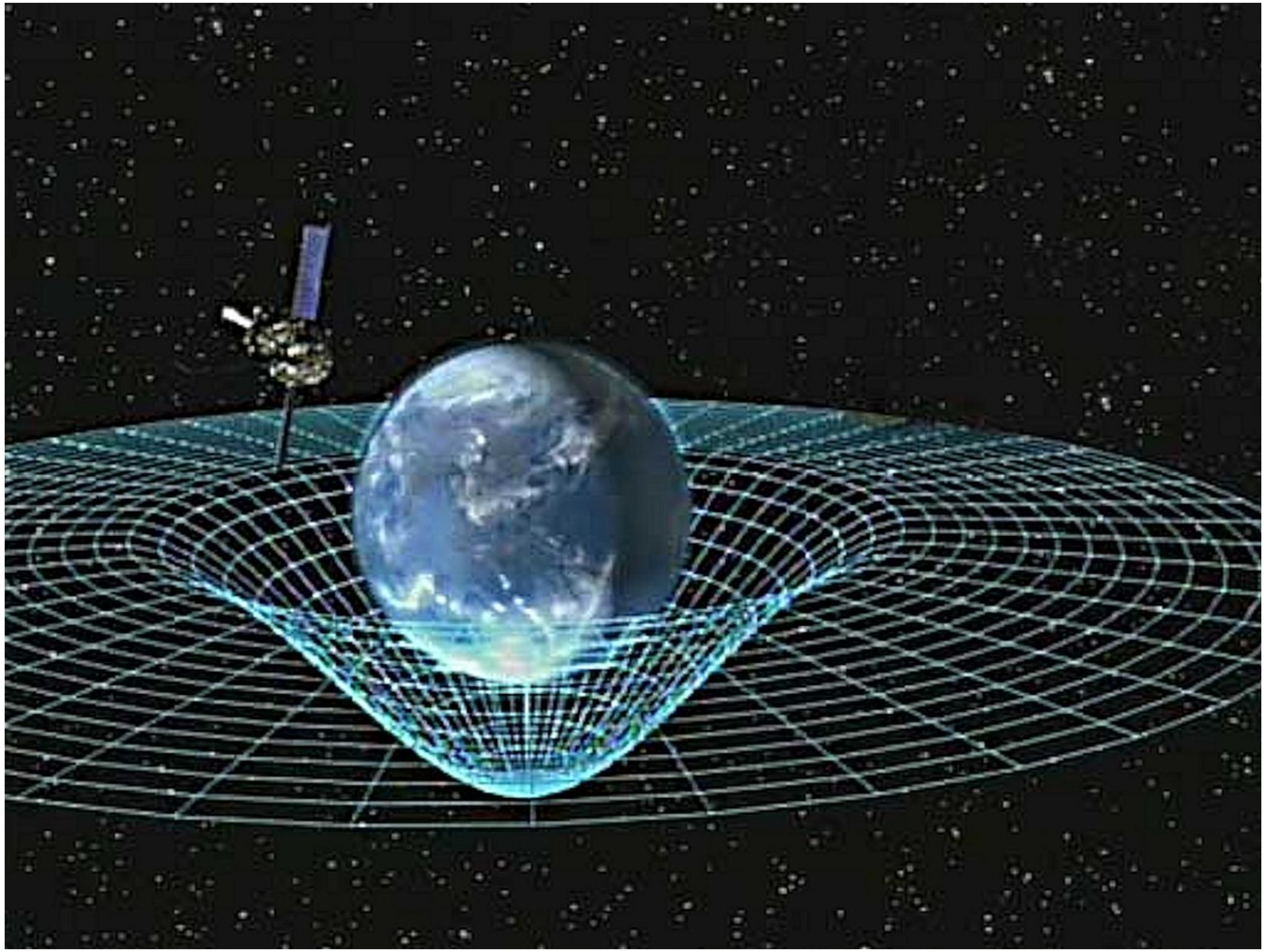

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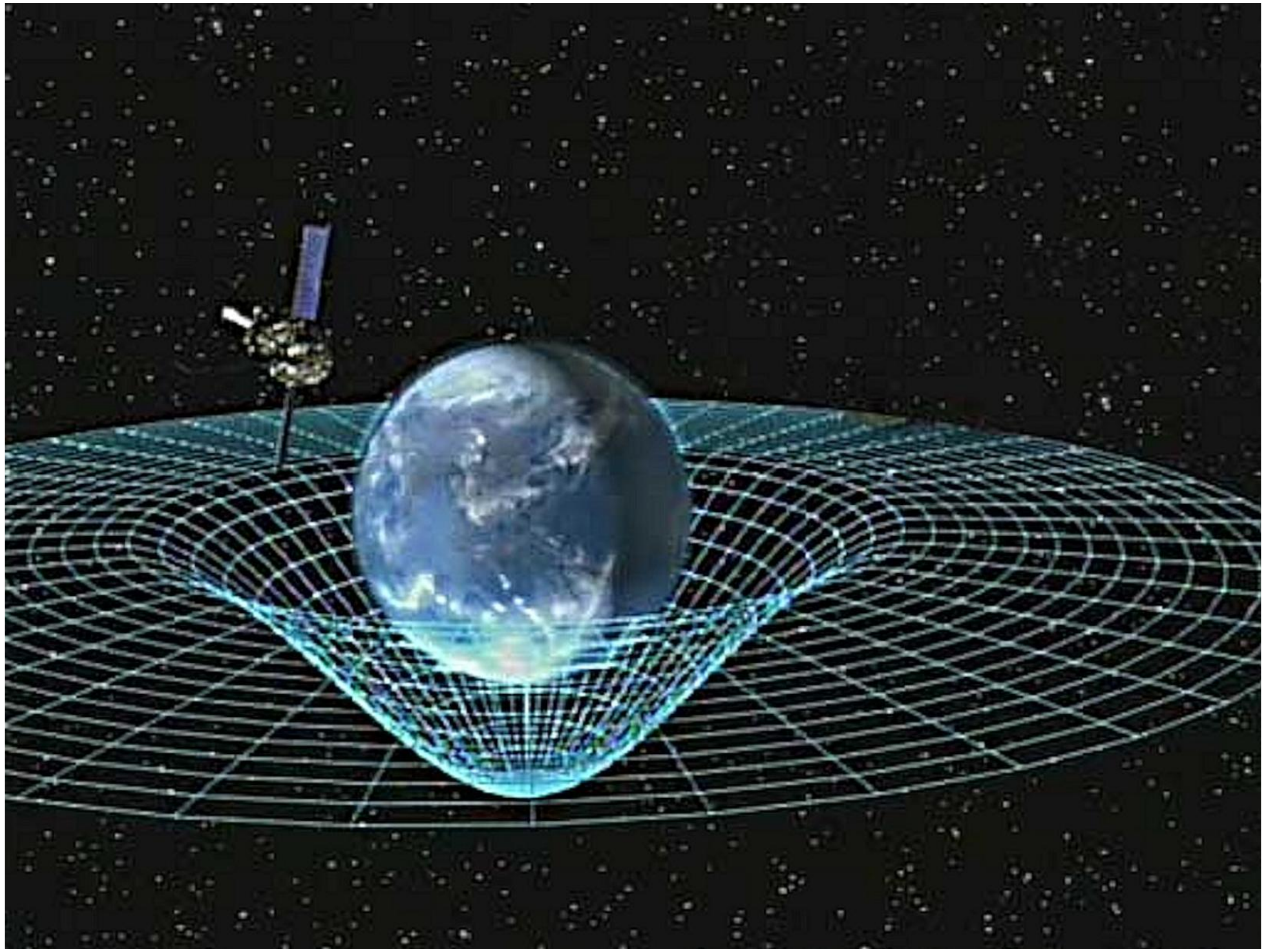

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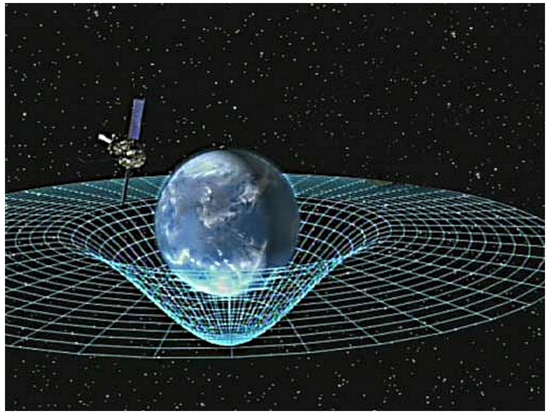
To describe a curved Universe, we need the concept of **manifold**, since there is no outside.

It makes no physical sense to embed a $4D$ spacetime into $\mathbb{R}^n$.

Although mathematically it is ok.

Riemannian geometry and general relativity



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Riemannian geometry and general relativity

General Relativity (1915)

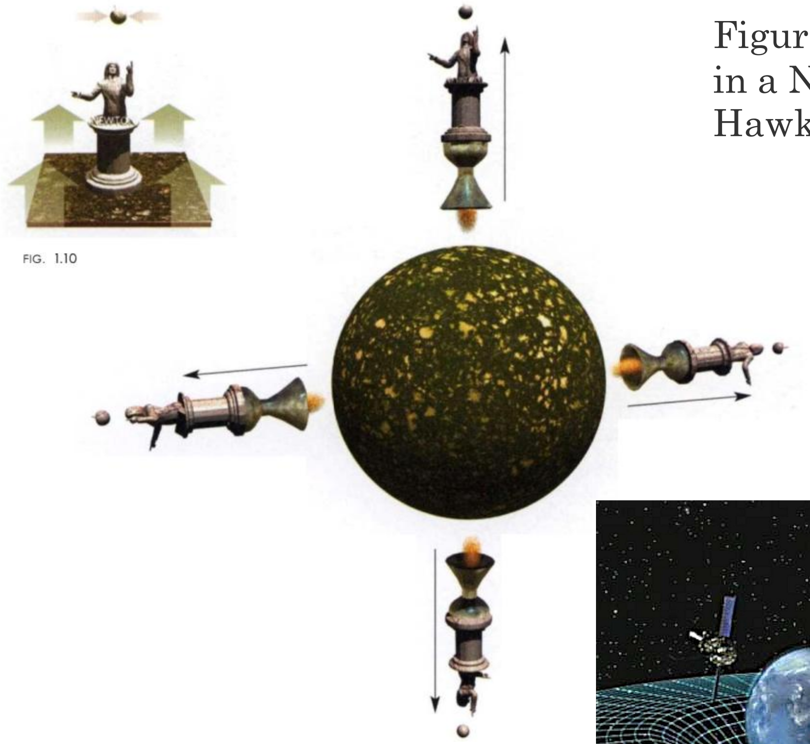
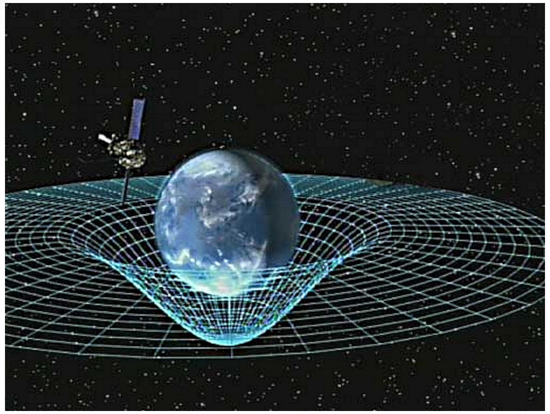


FIG. 1.10

Figure: The Universe in a Nutshell, Stephen Hawking.



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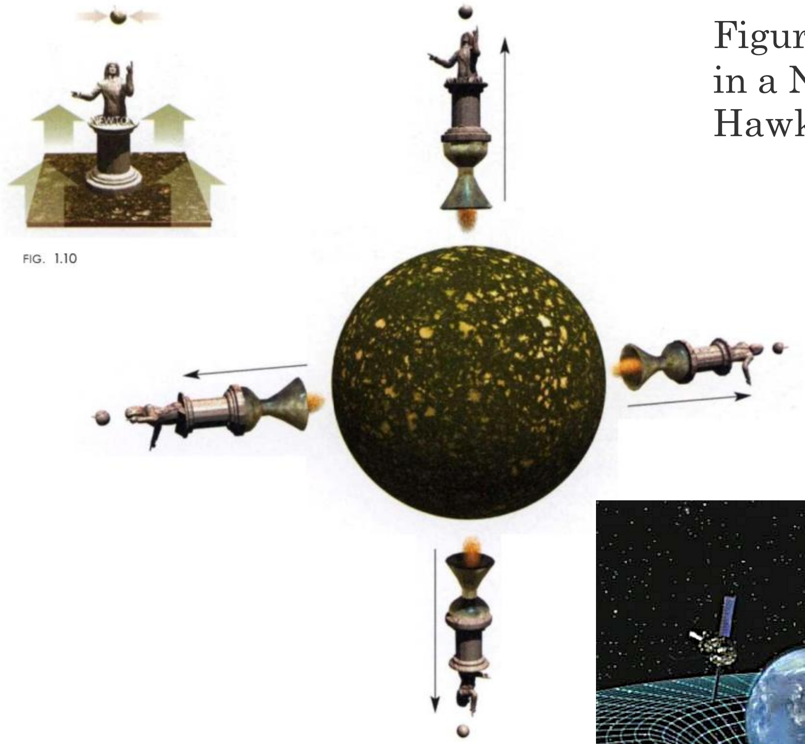

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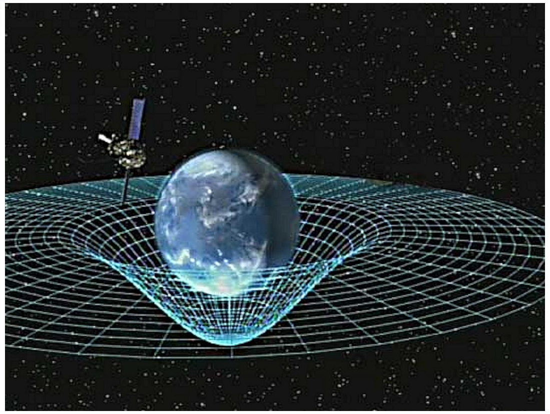

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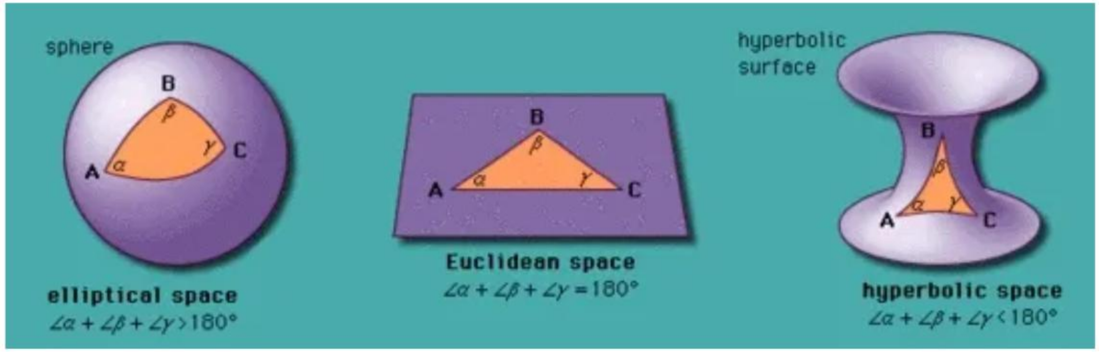

Figure: https://medium.com/@Merrysci/riemanns-novel-lecture-that-changed-the-course-of-geometry-50db478159b8

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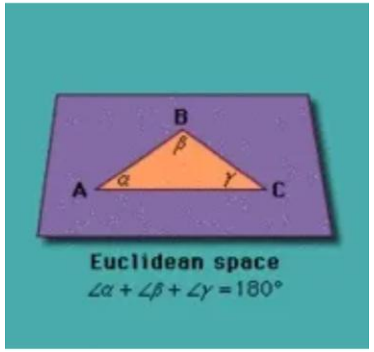

Euclidean Geometry (300 BCE):

1) Line Segment Connection
2) Extension
3) Circle Construction
4) Congruence of Right Angles
5) The Parallel Postulate

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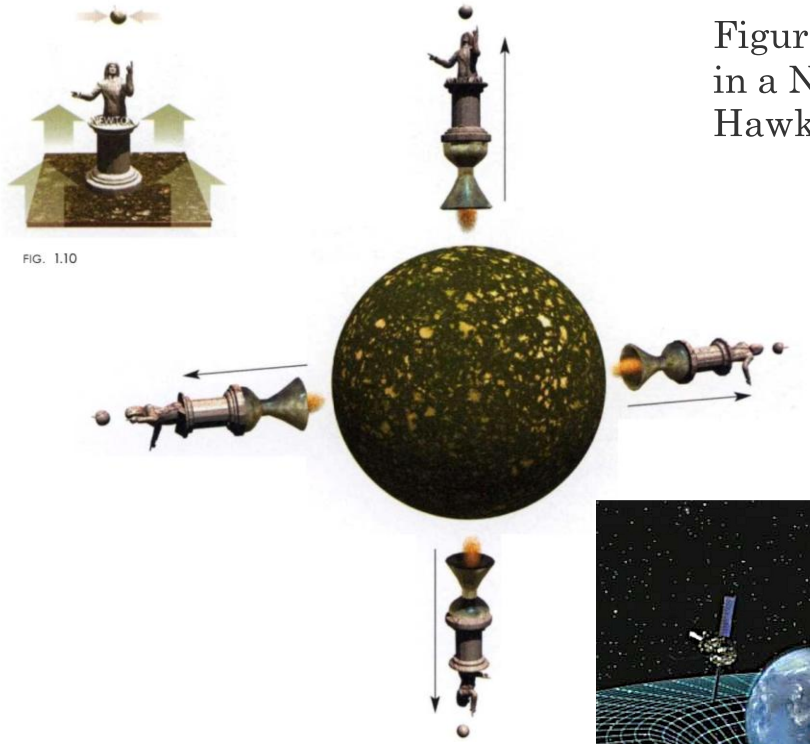

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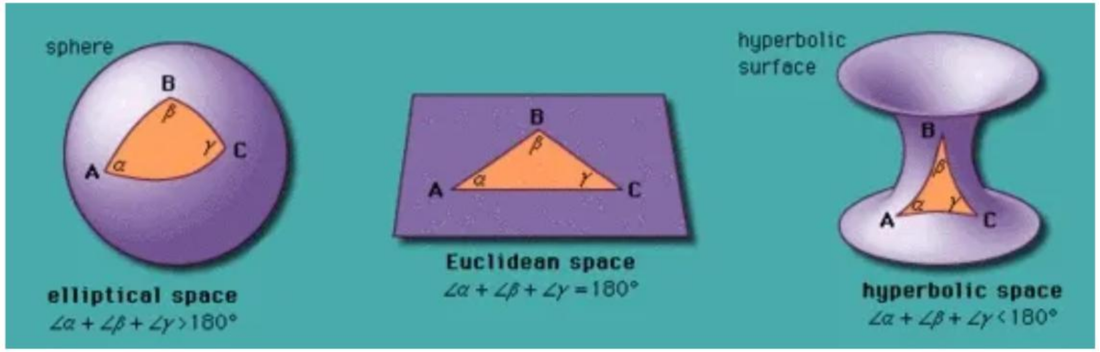

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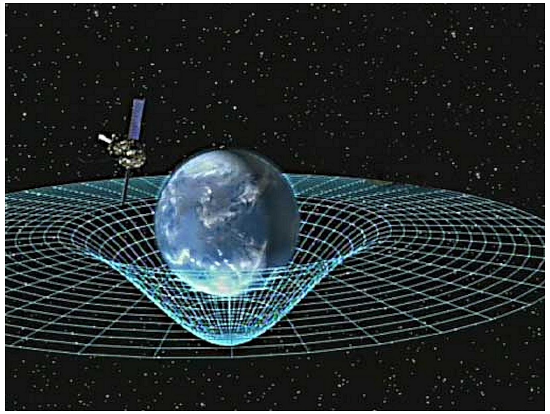

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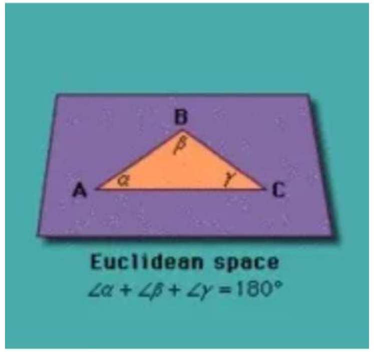

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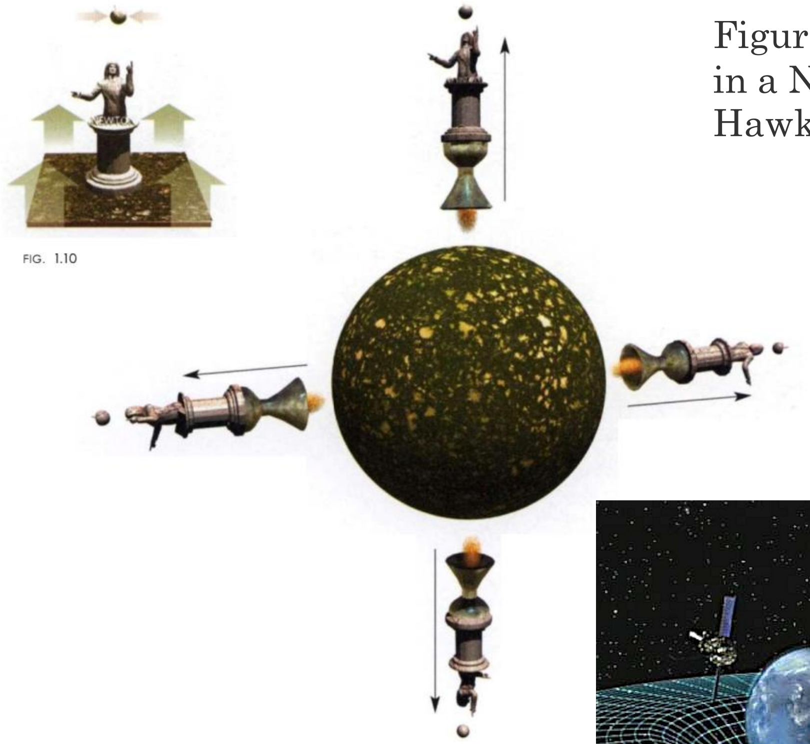

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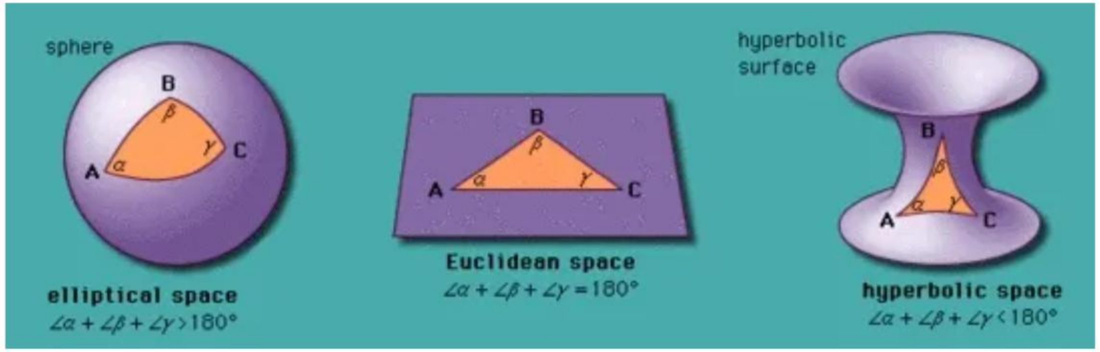

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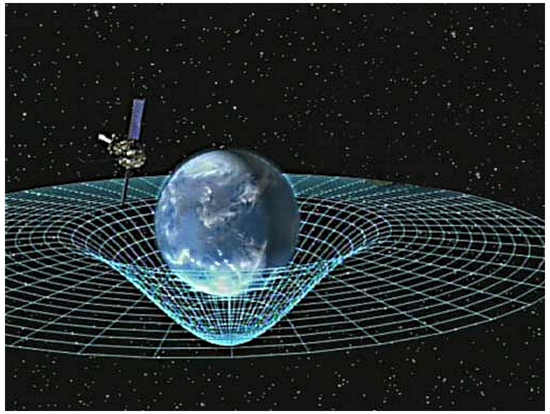

Figure: Wikipedia.

Cold atom labs in space and on the ground

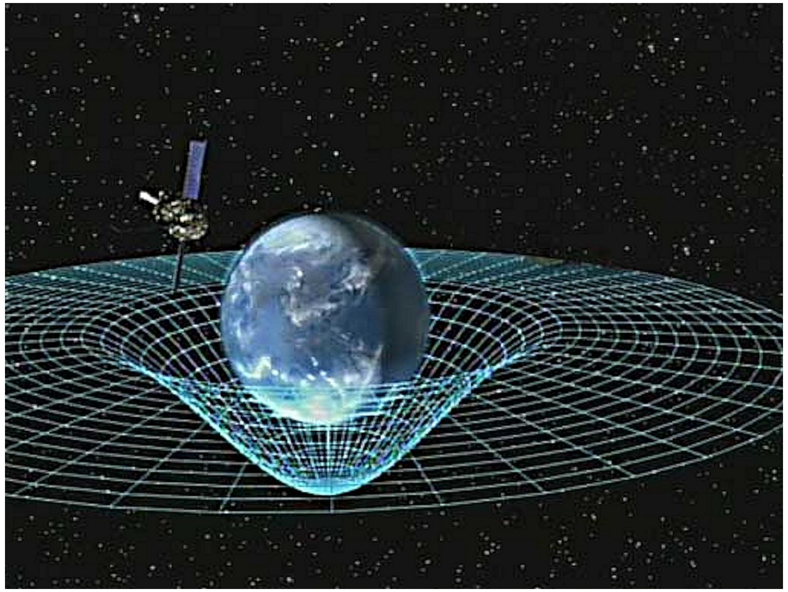

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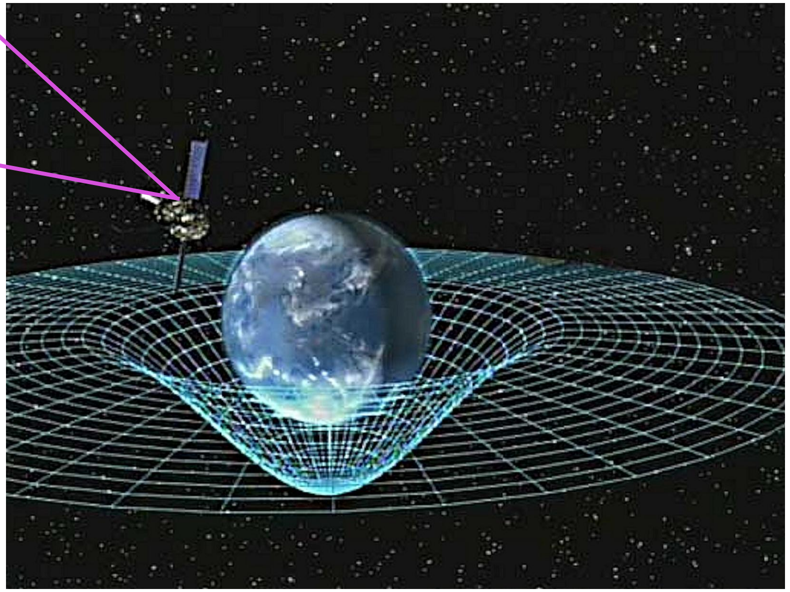
Figure: https://issnationallab.org/facilities/cold-atom-lab/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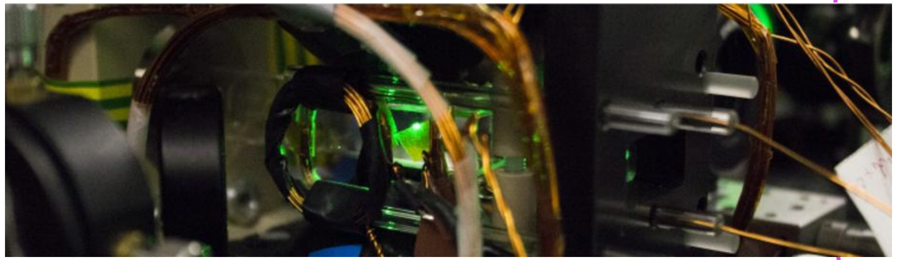
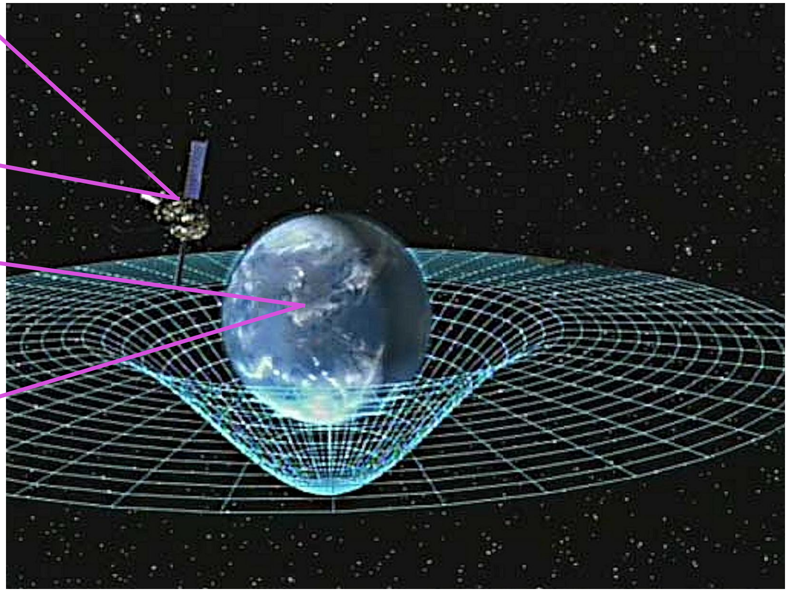


Figure: https://bec.lpl.univ-paris13.fr/Index.htm



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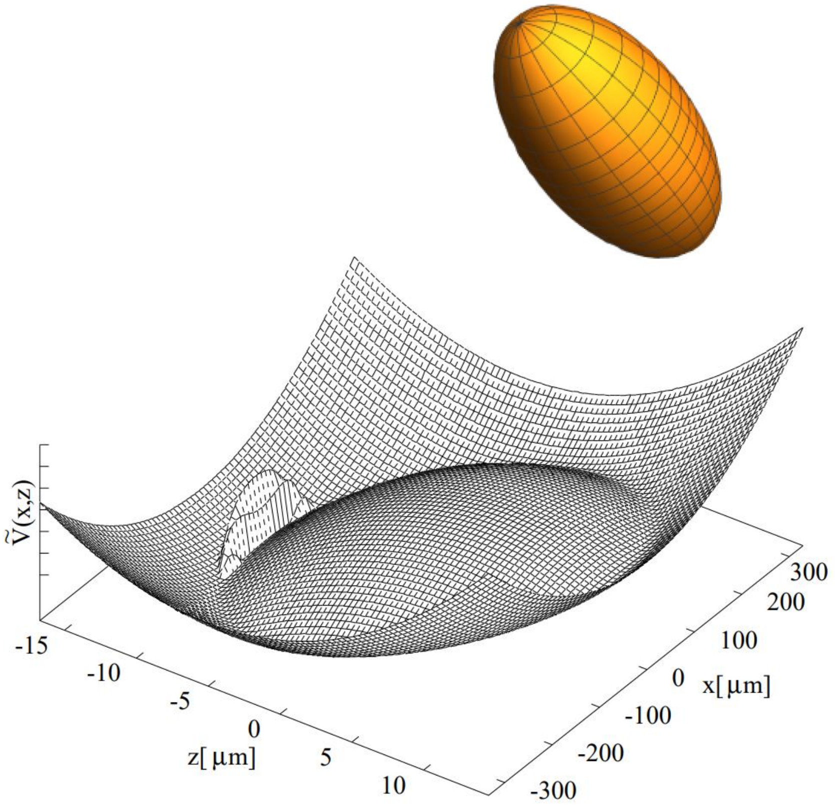
BEC on a bubble trap

Theoretical proposal (2001):



O. Zobay and B. M. Garraway, PRL **86** 1195 (2001)

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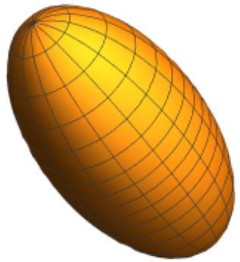

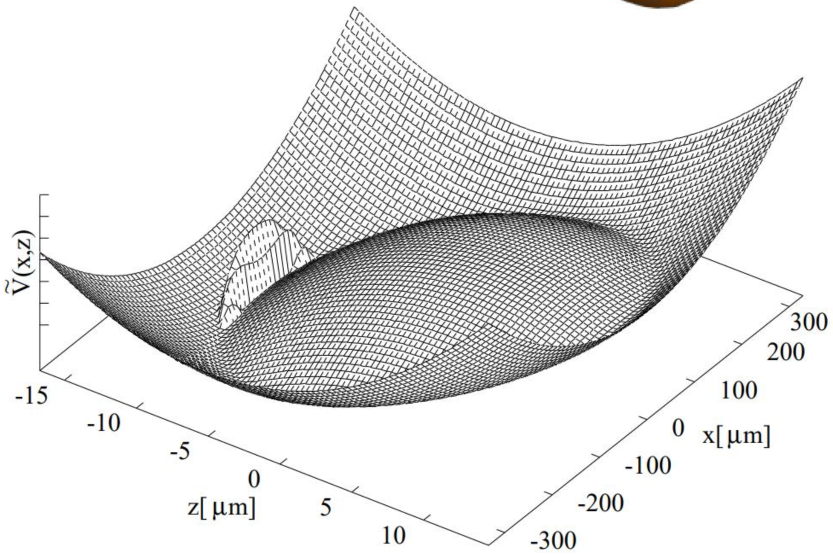

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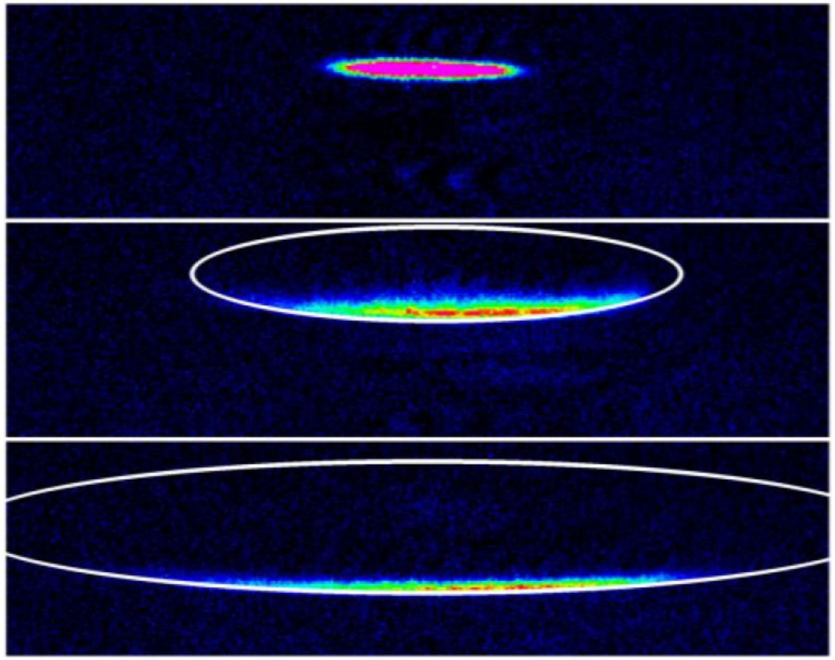
First experiments (2004):



Y. Colombe, B. Mercier, H. Perrin, and V. Lorent,
J. Phys. IV 116, 247 (2004).

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and H. Perrin, EPL **67** 593 (2004)

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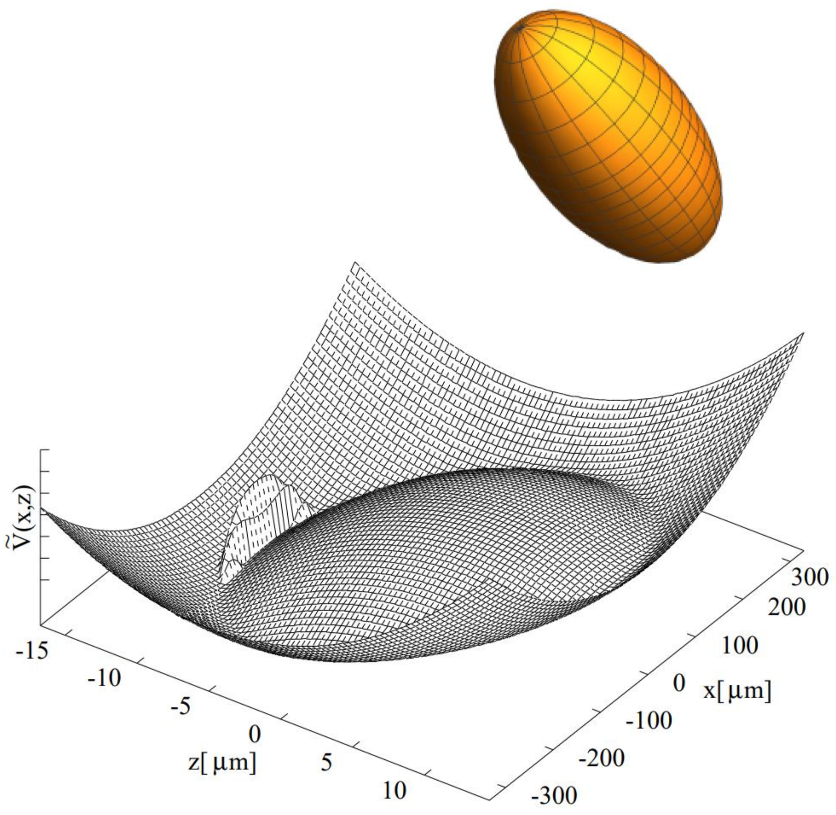

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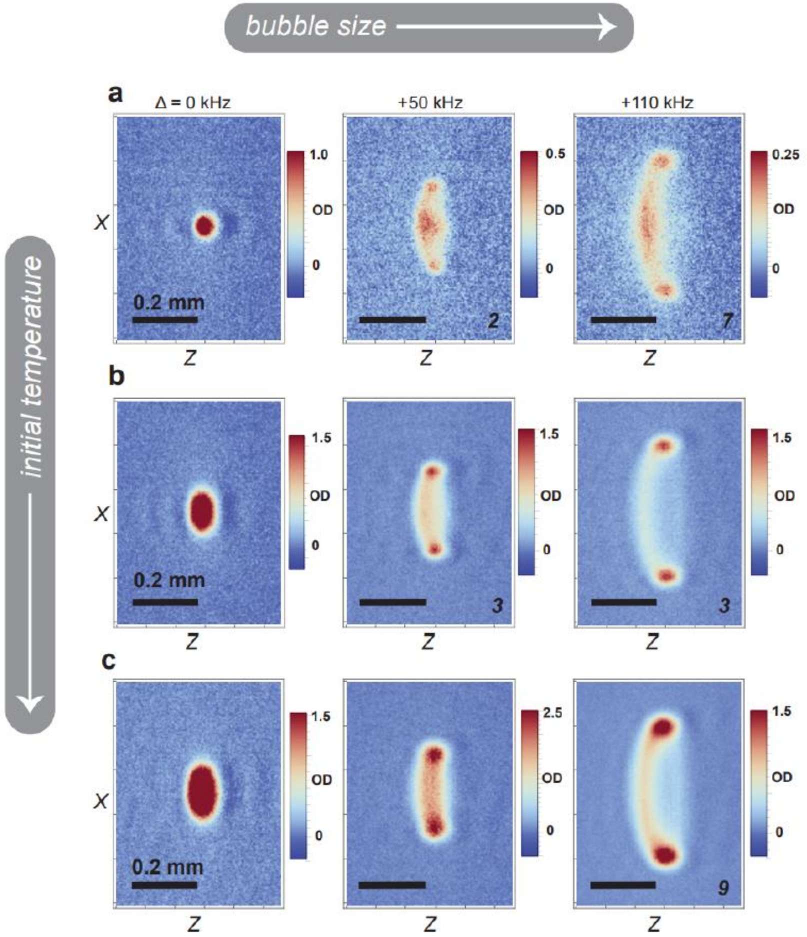
Recent experiments in space (2021):



R. A. Carollo, D. C. Aveline, B. Rhyno, S. Vishveshwara, C. Lannert, J. D. Murphree, E. R. Elliott, J. R. Williams, R. J. Thompson, N. Lundblad, Nature **606**, 281 (2022)

Recapitulation

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Surface

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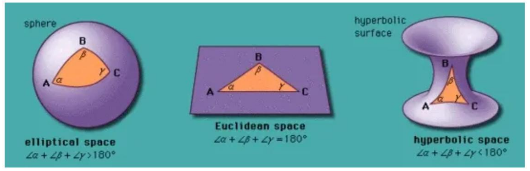

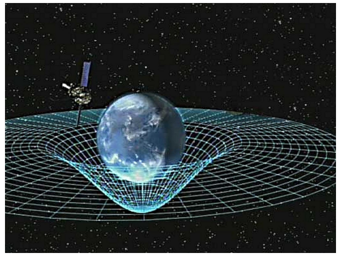

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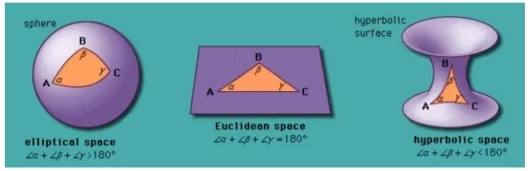

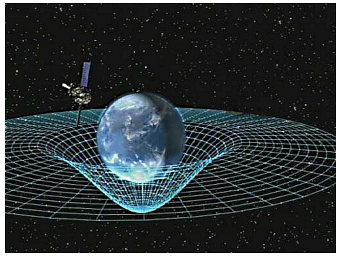

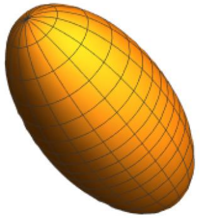

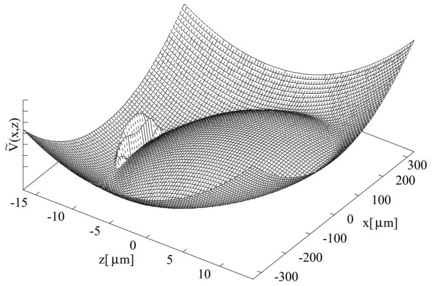

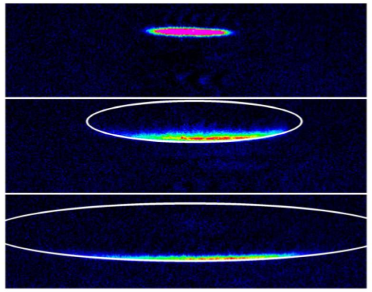

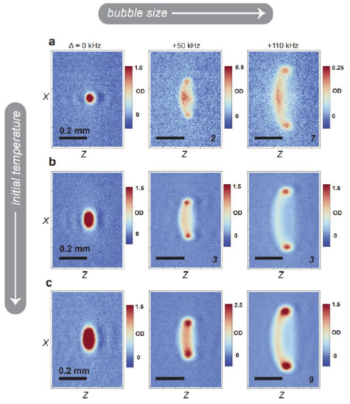

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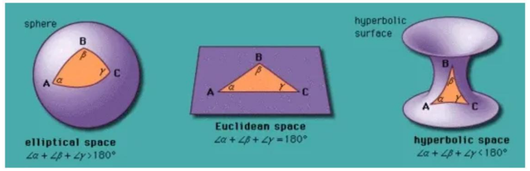

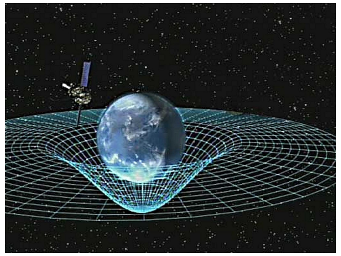

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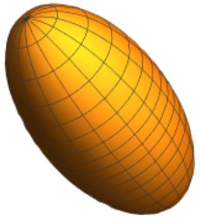
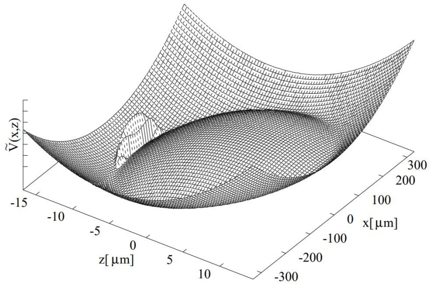
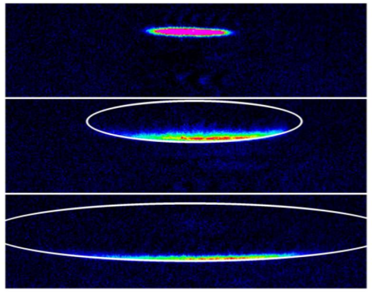
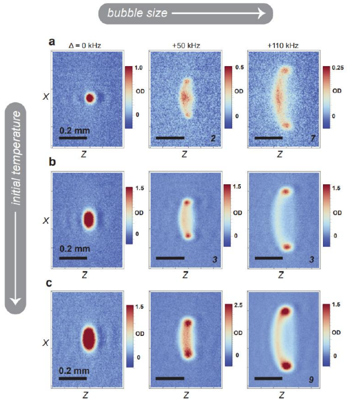


Can the geometry affect the physics of a system?







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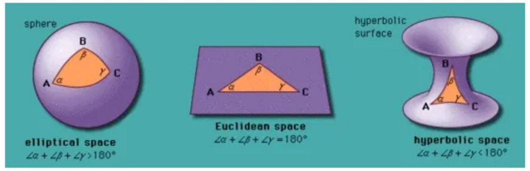

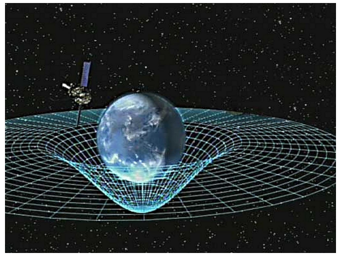

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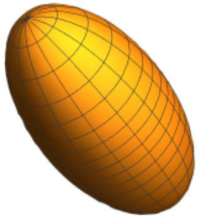

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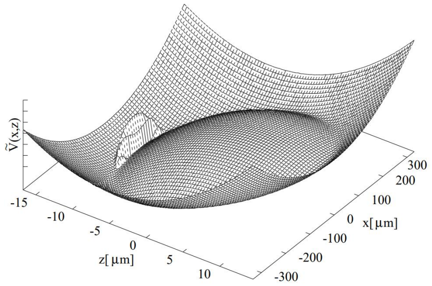

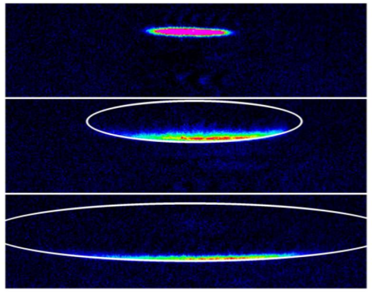

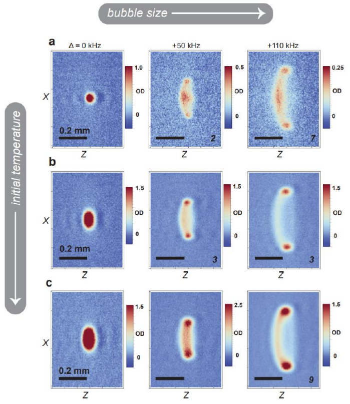

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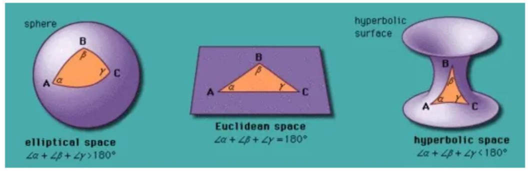

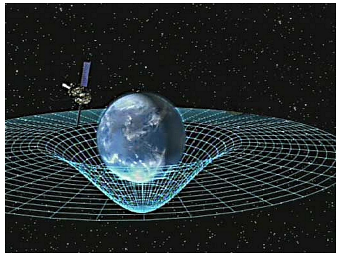

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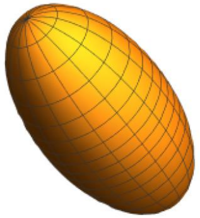

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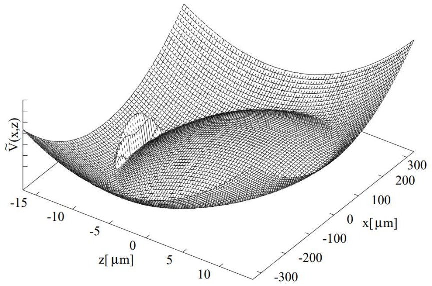

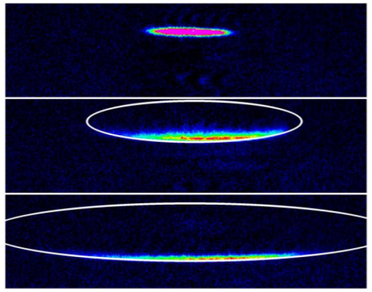

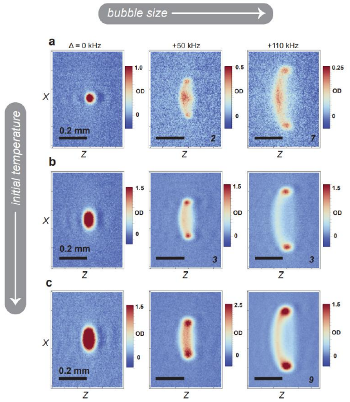

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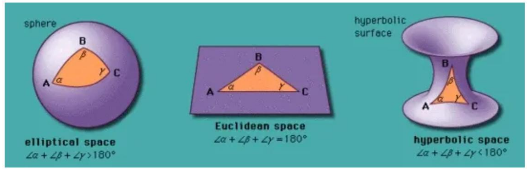

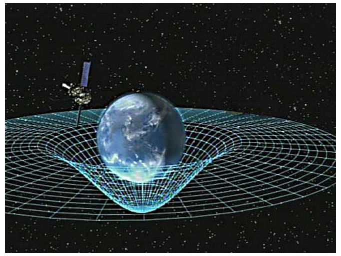

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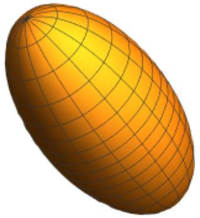

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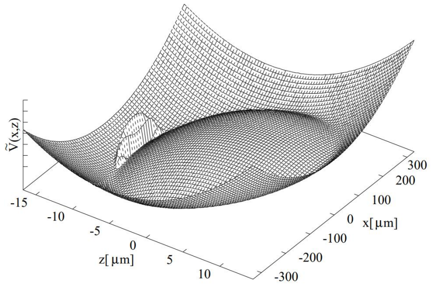

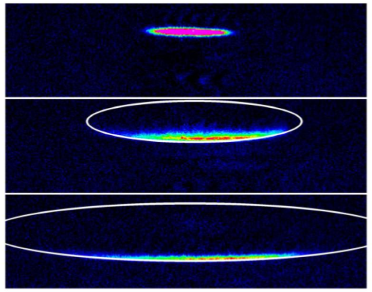

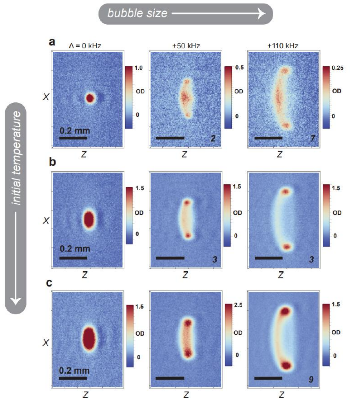

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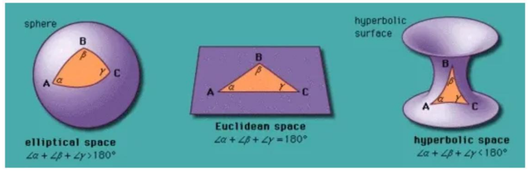

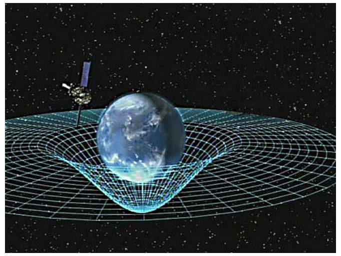

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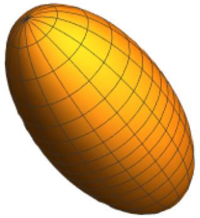

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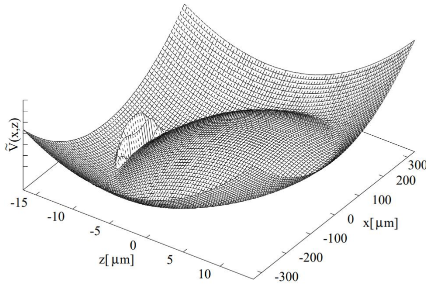

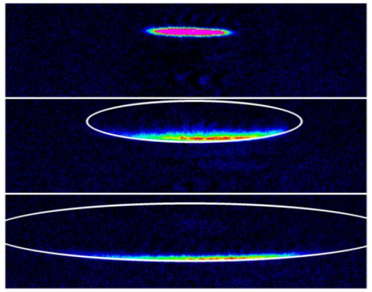

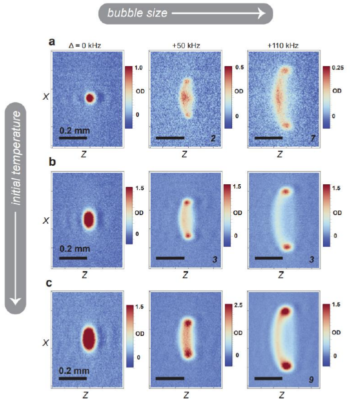

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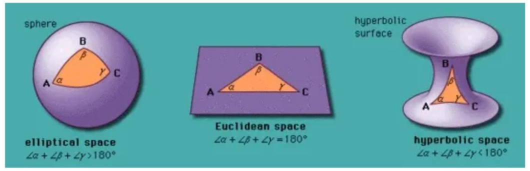

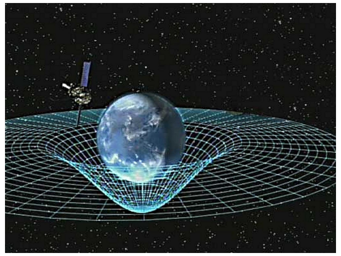

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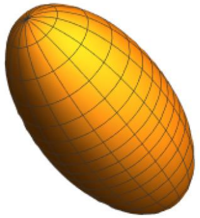

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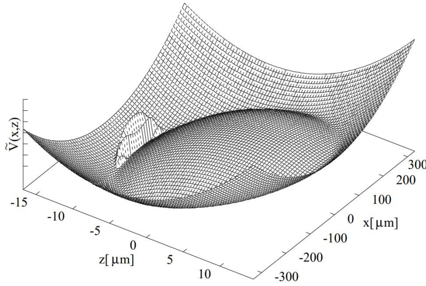

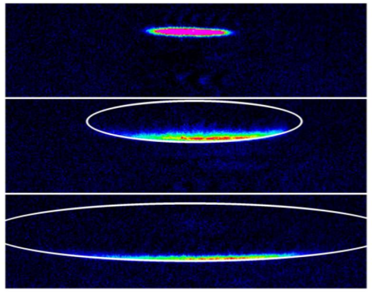

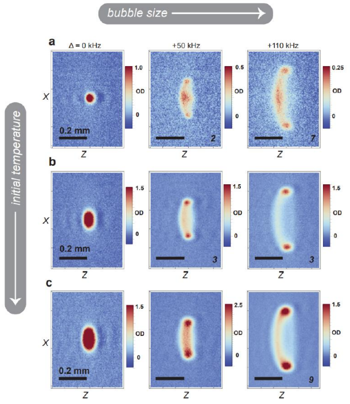

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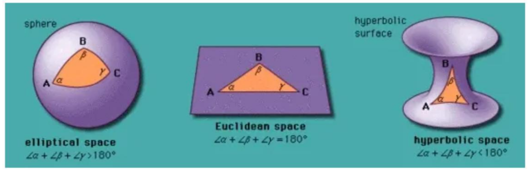

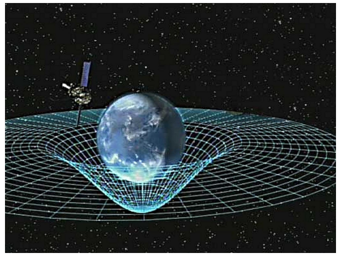

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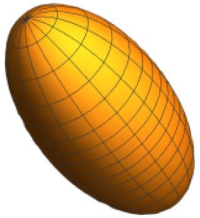

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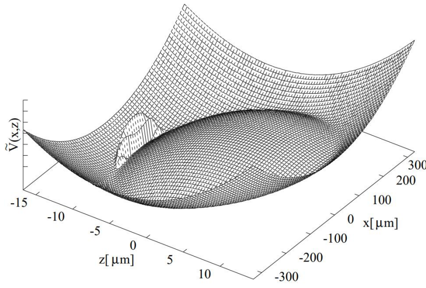

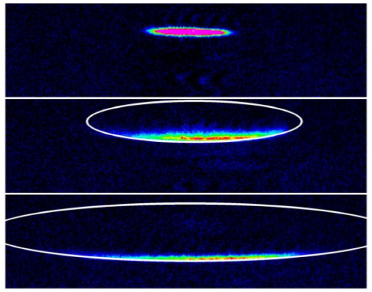

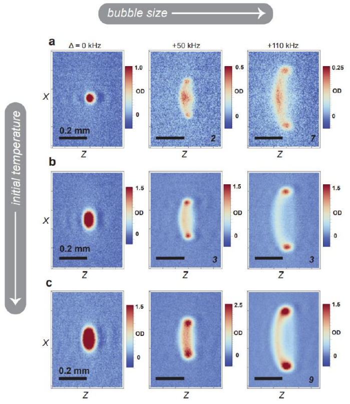

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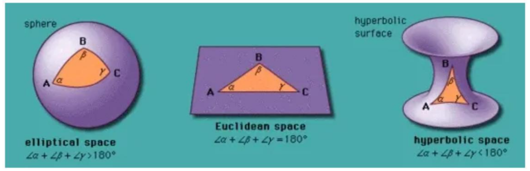

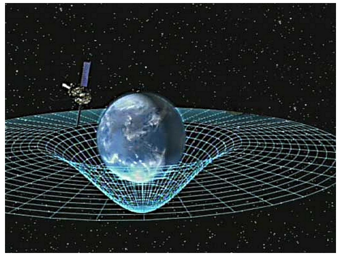

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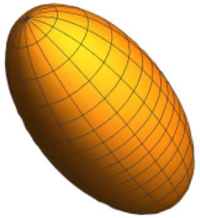

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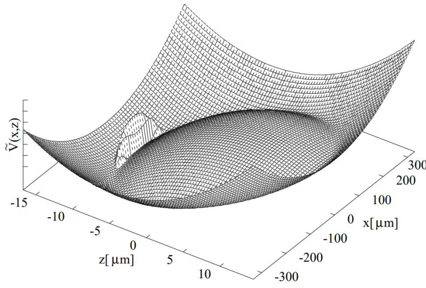

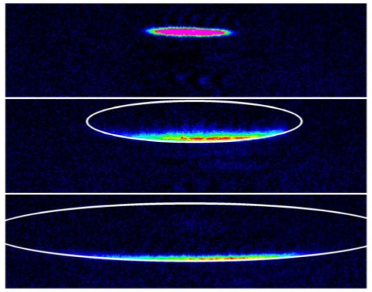

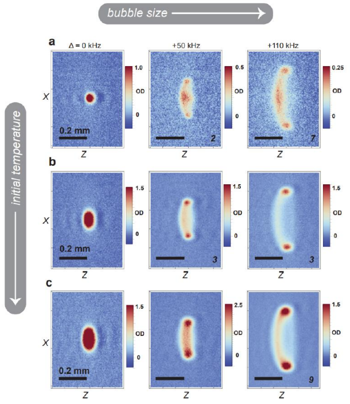

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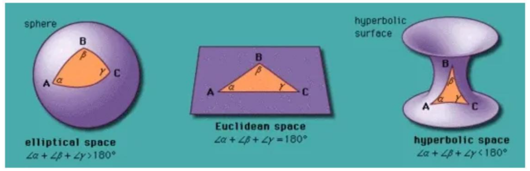

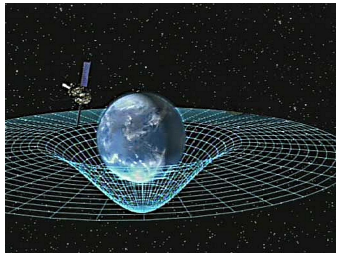

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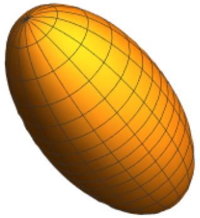

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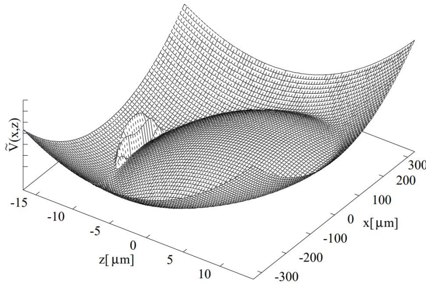

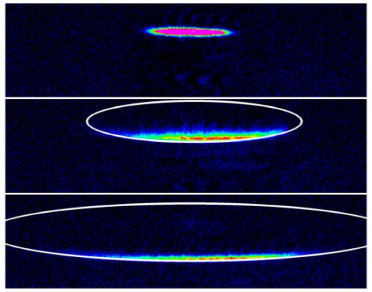

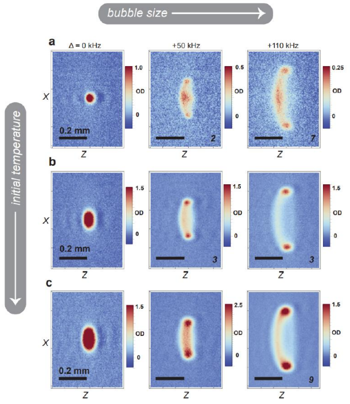

Classical and quantum systems on curved surfaces

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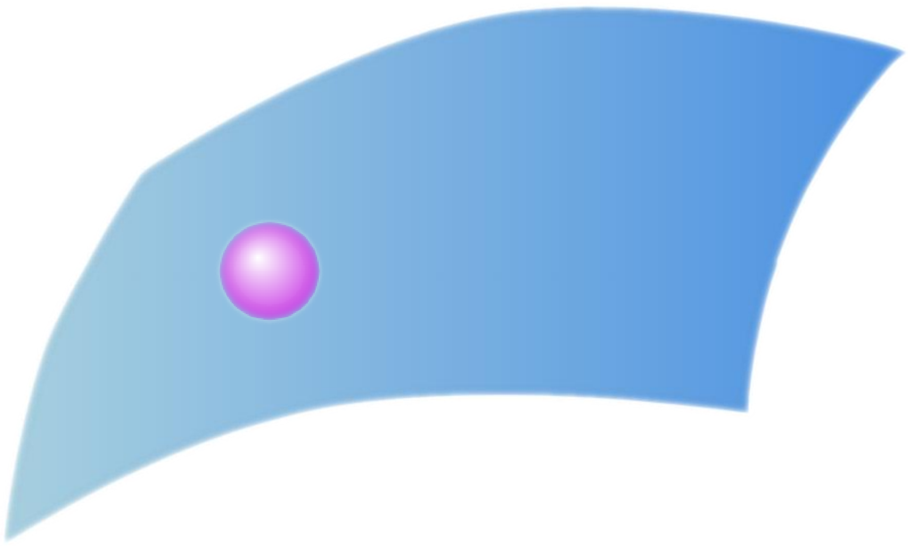

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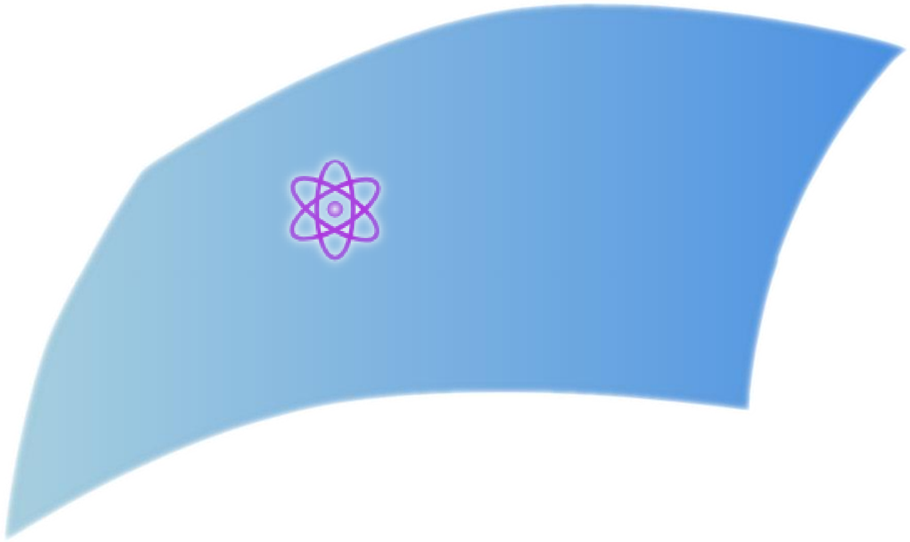

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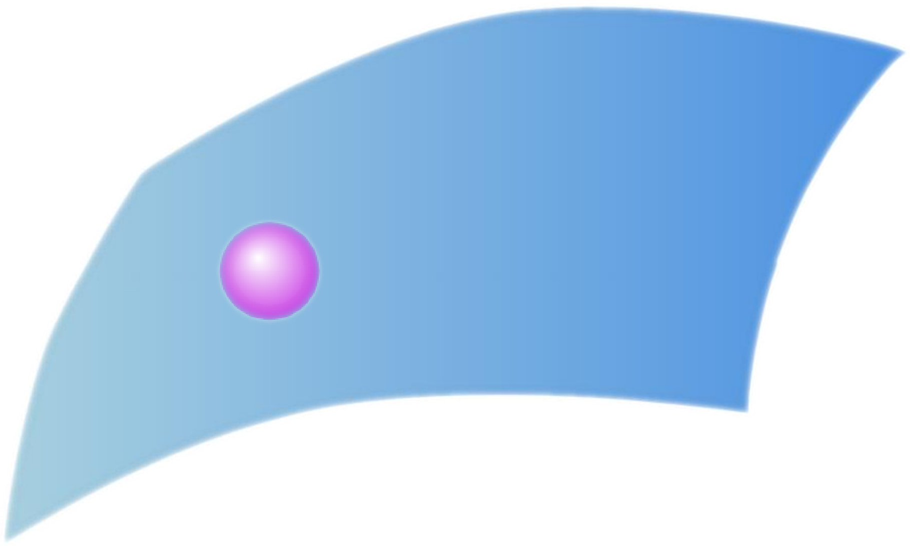

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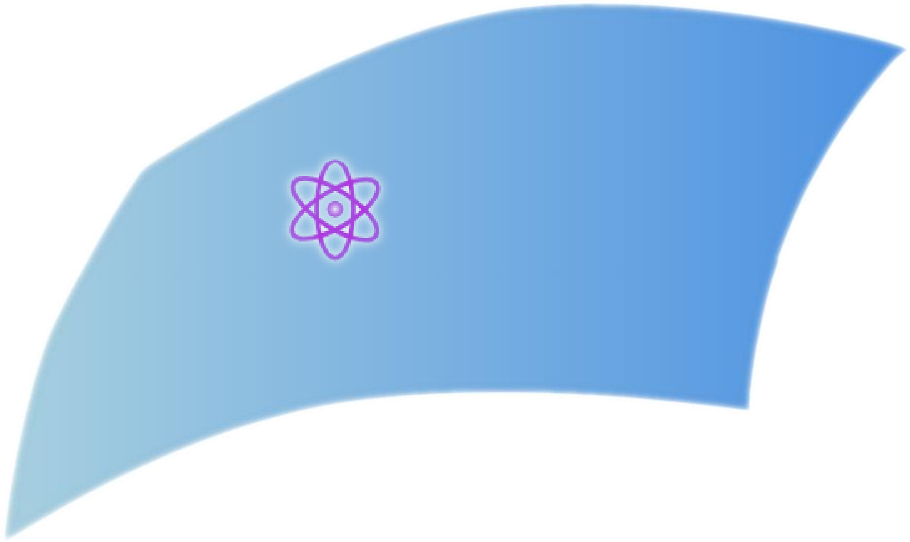

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Marion and Thornton, Classical Dynamics.

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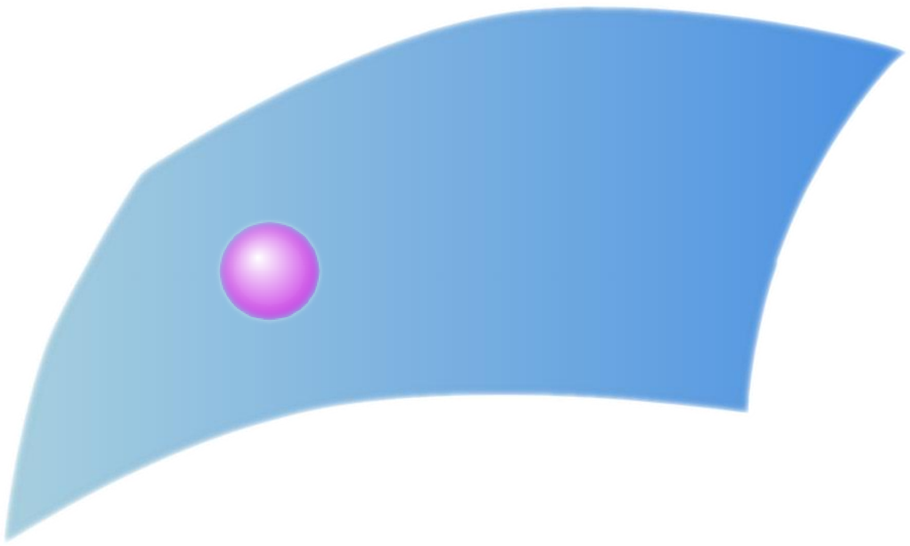

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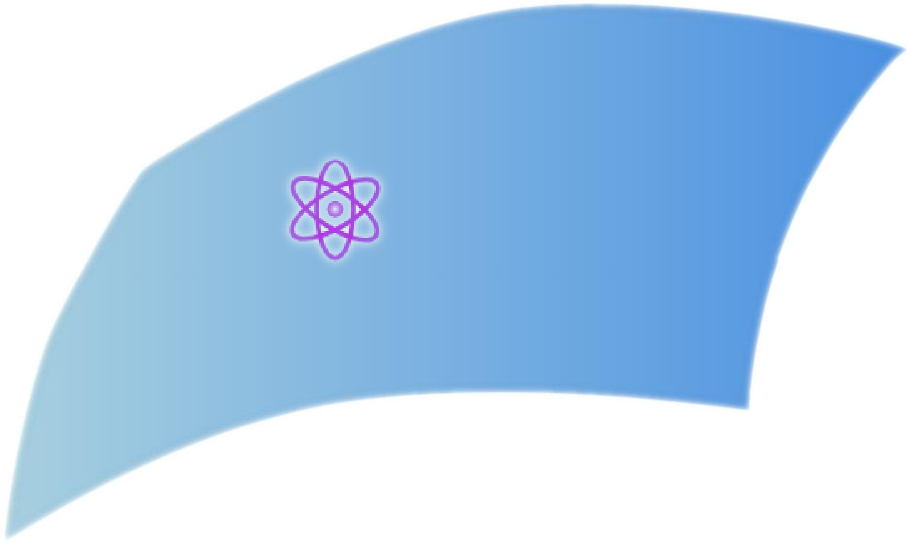

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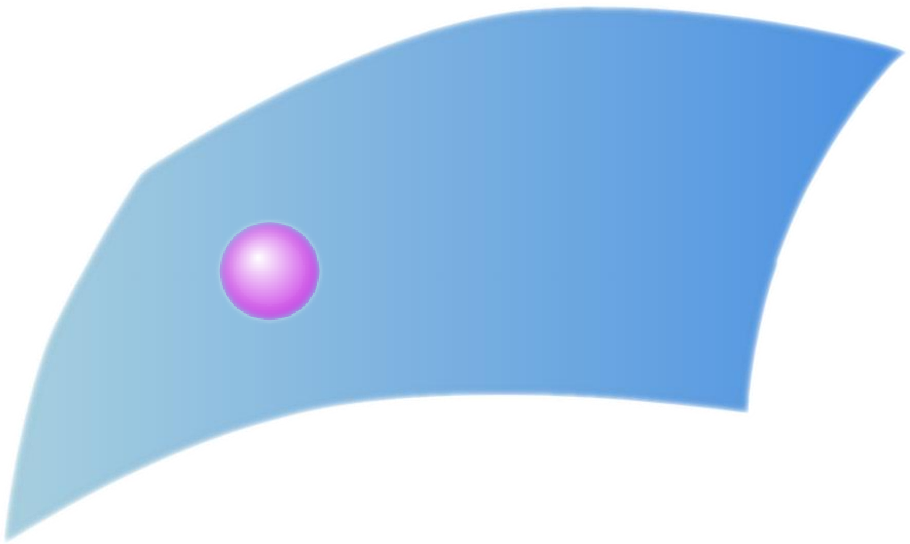

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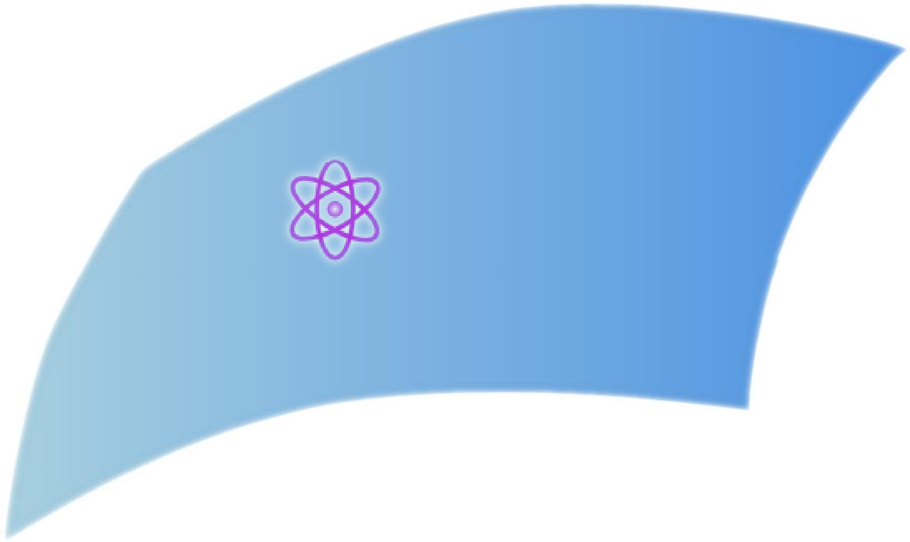

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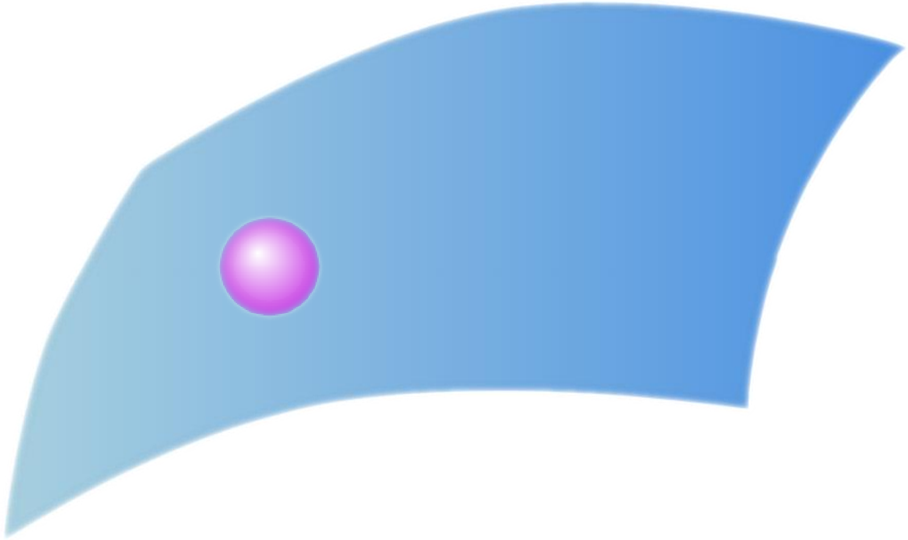

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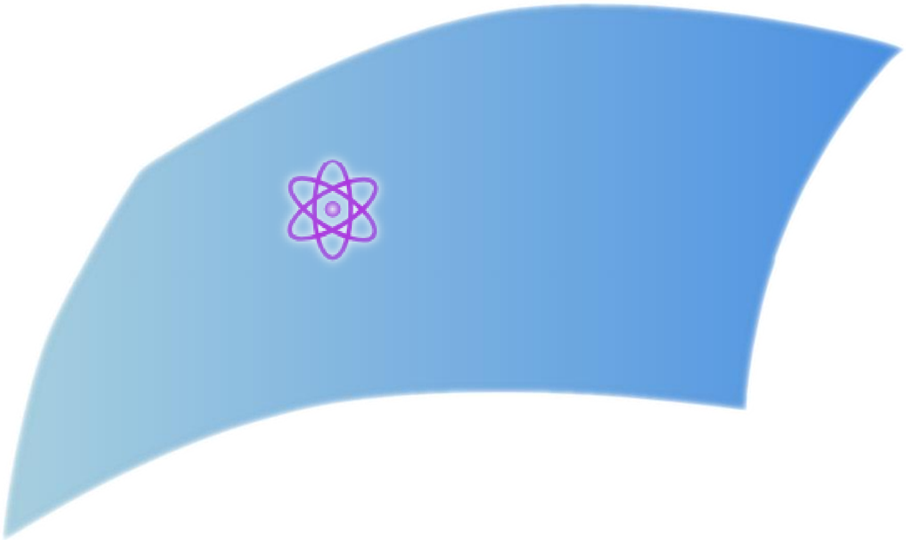

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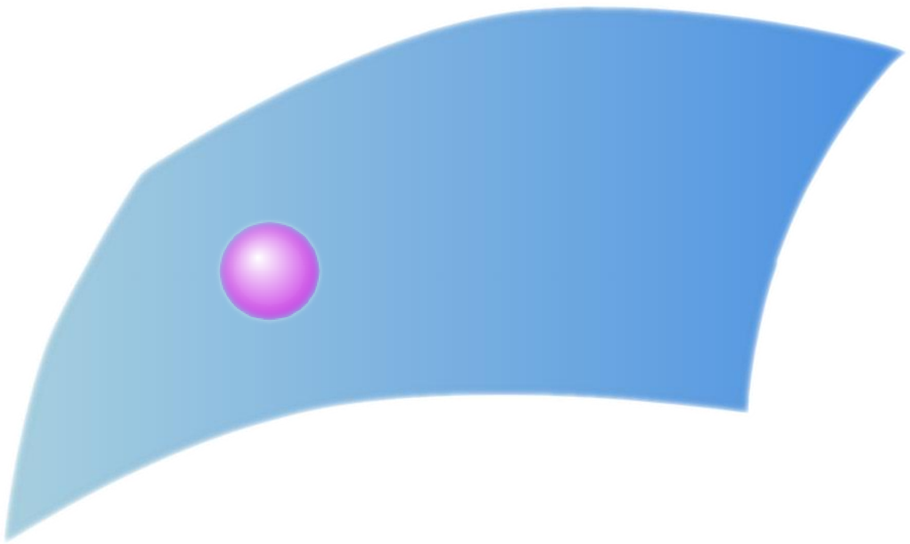

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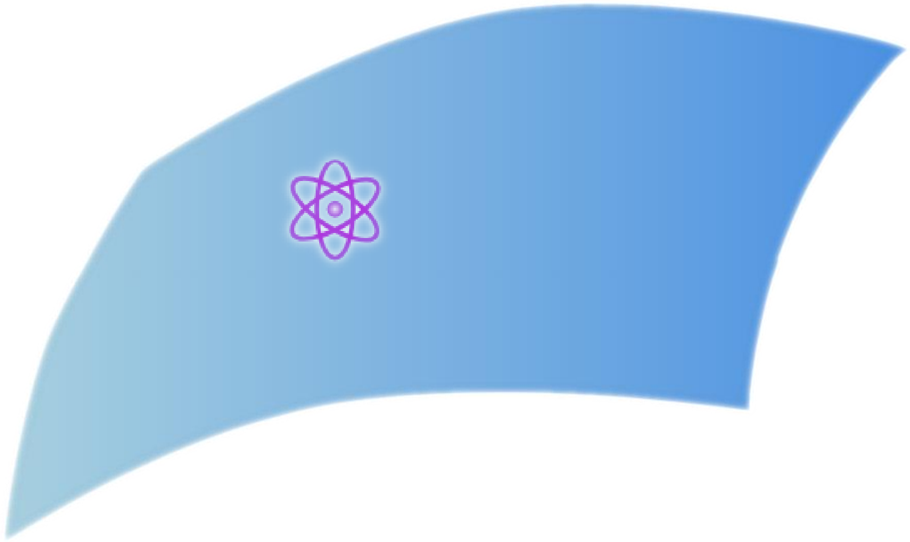

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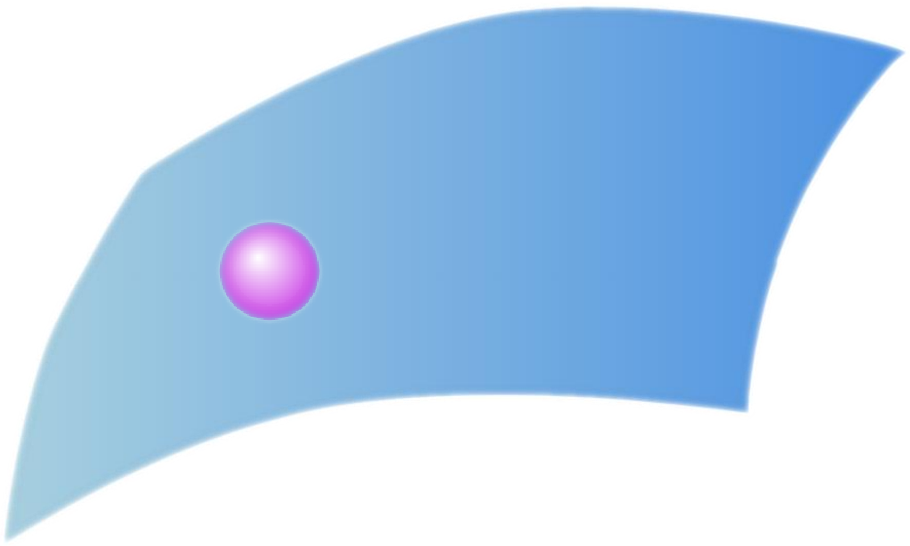

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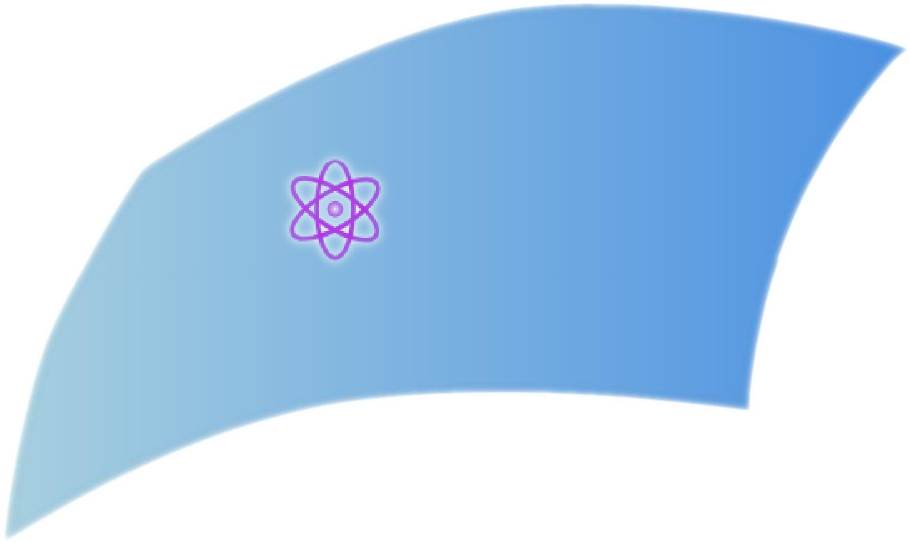

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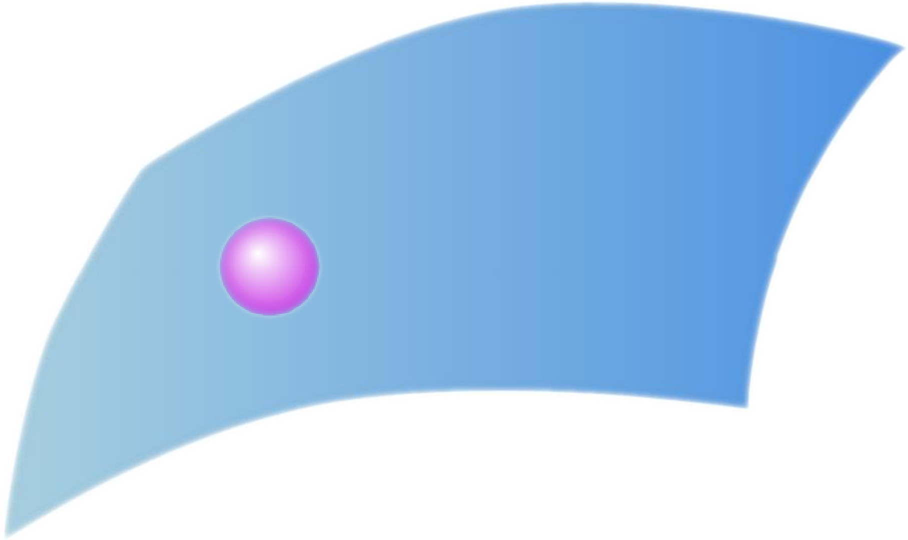


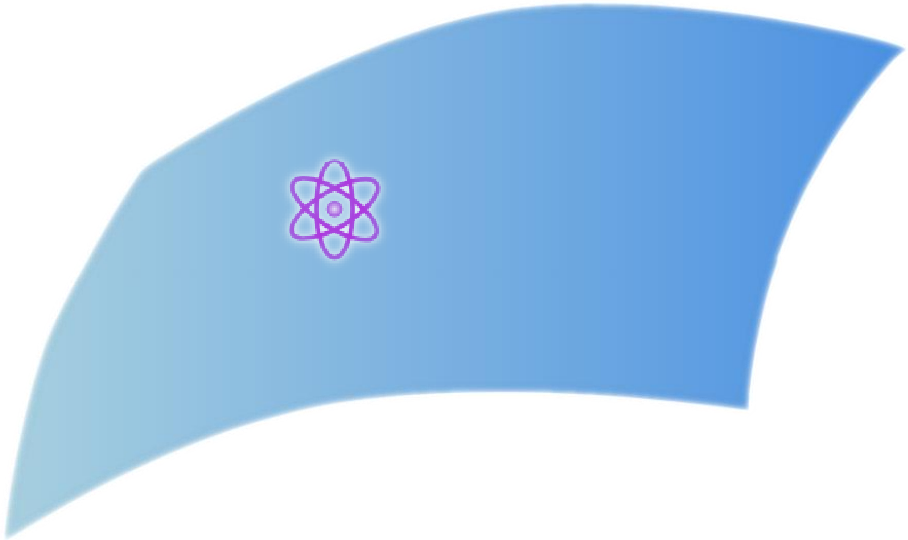
Newton's laws predict normal force: $\boldsymbol{F} = mv^2\kappa_t\boldsymbol{n}$

They are equivalent to the Lagrangian formulation.

Easier to use Lagrangian formulation: fewer variables.

Marion and Thornton, Classical Dynamics.

Quantum system on a surface:



- Bryce S. DeWitt, Rev. Mod. Phys. 29, 377 (1957);
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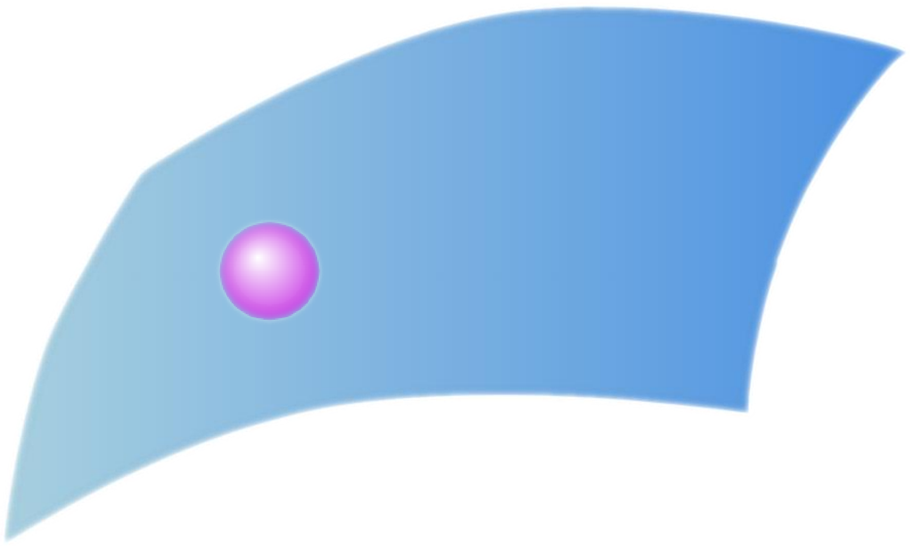

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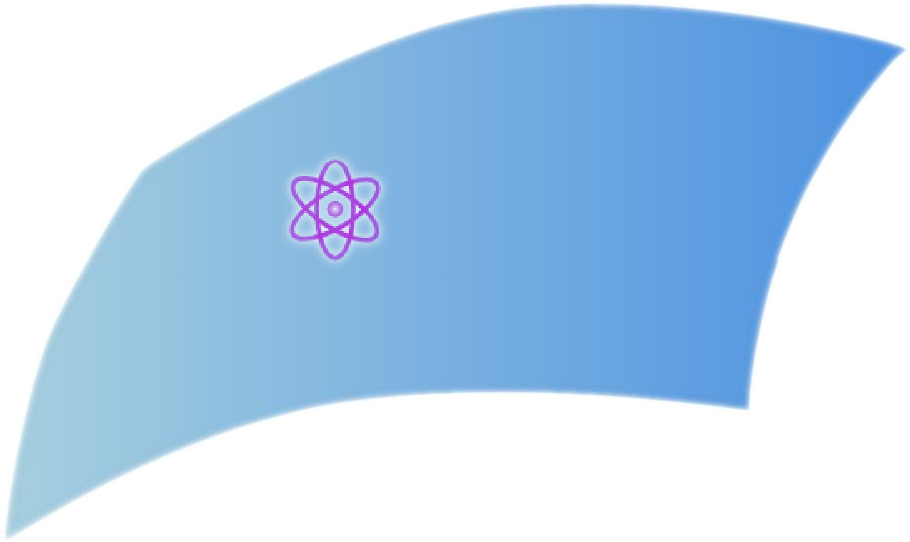

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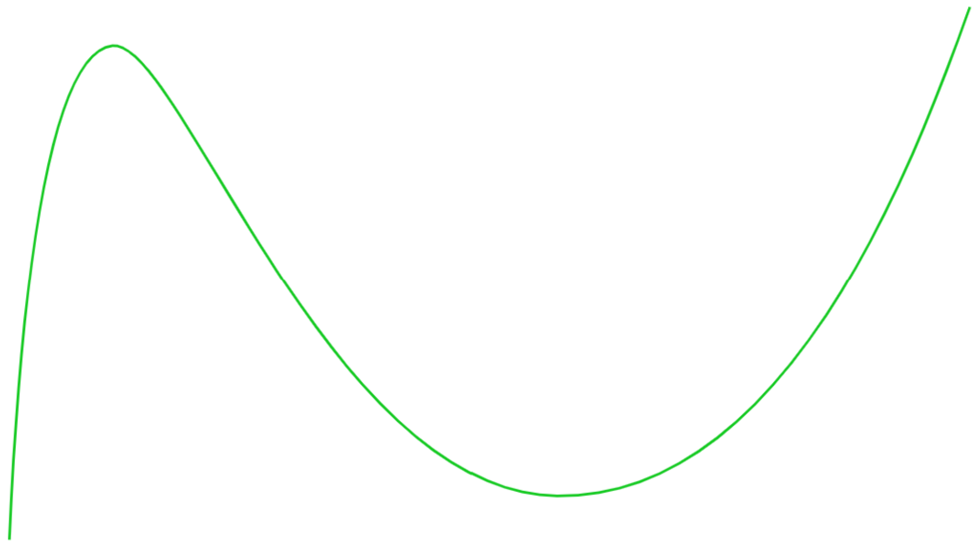
Curvature – 1D

Curvature of a curve:



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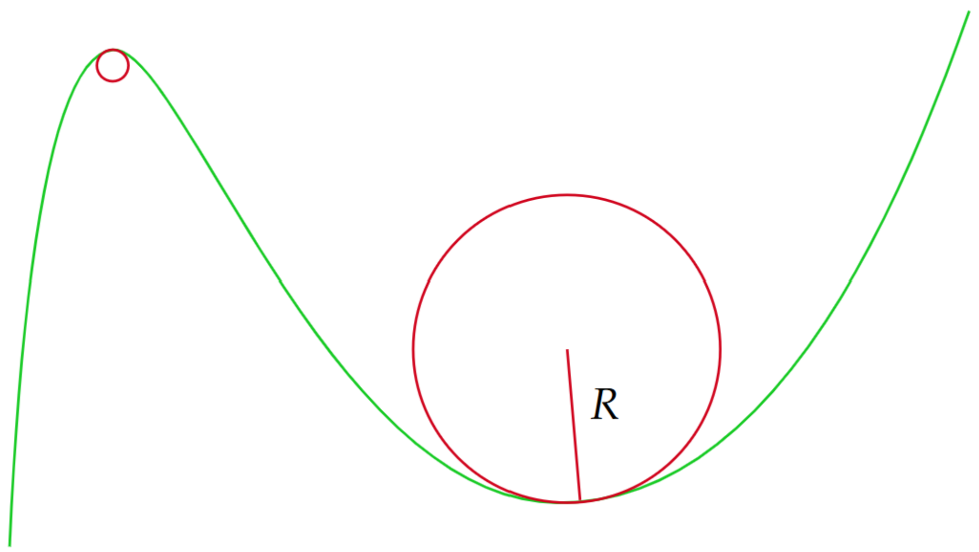

Figure based on: https://math.libretexts.org/

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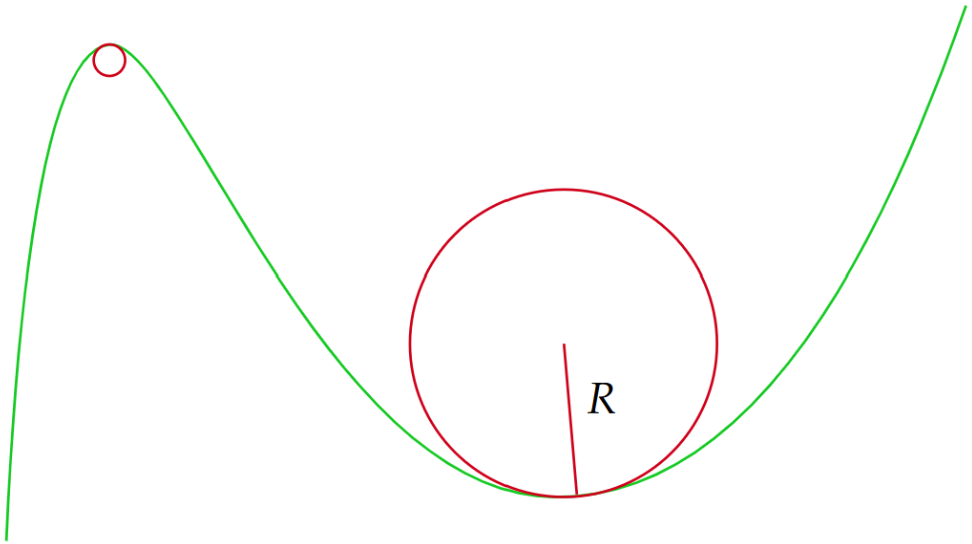
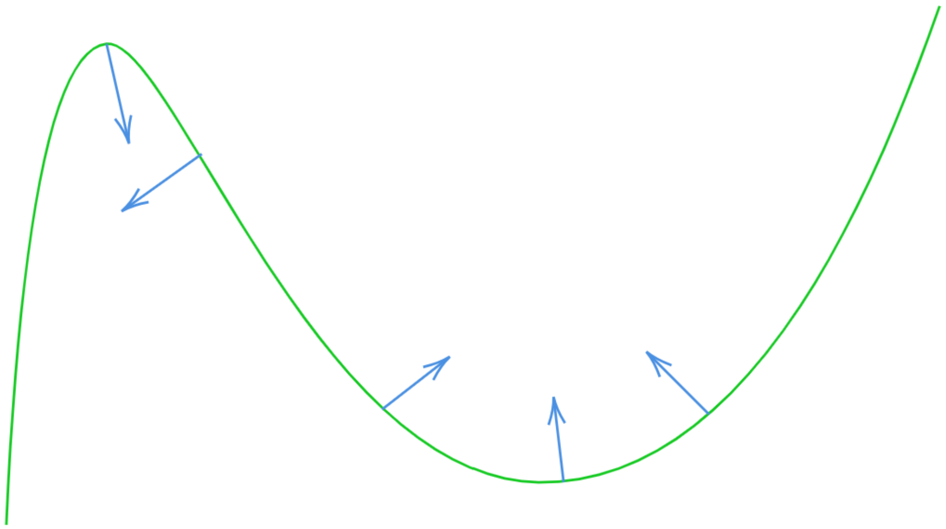


Curvature measures how fast the normal vector changes:



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Principal (extrinsic) curvatures – 2D

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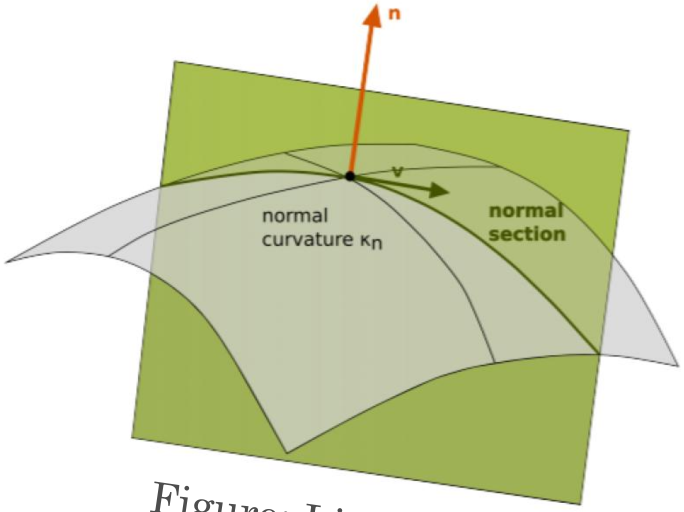

Figure: Lia Vas' notes

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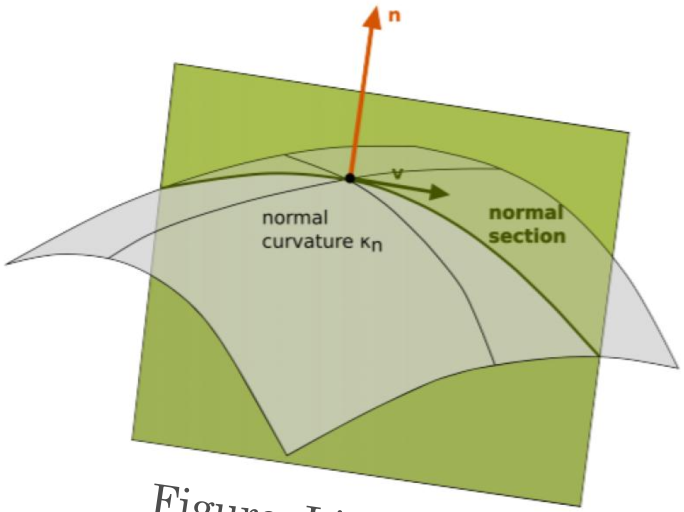

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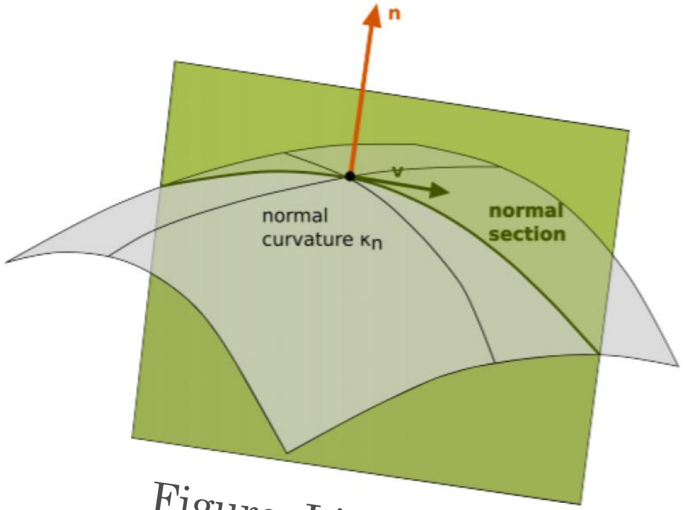

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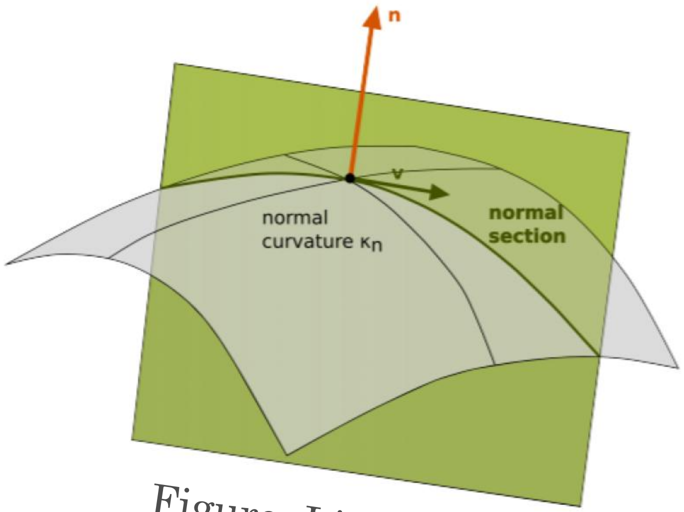

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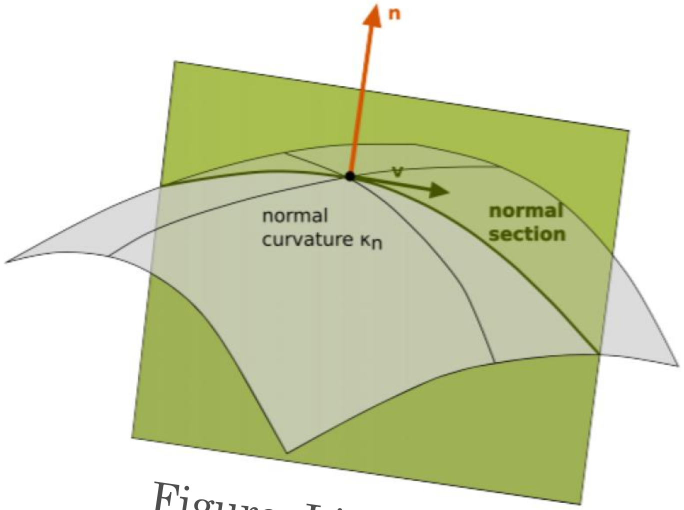

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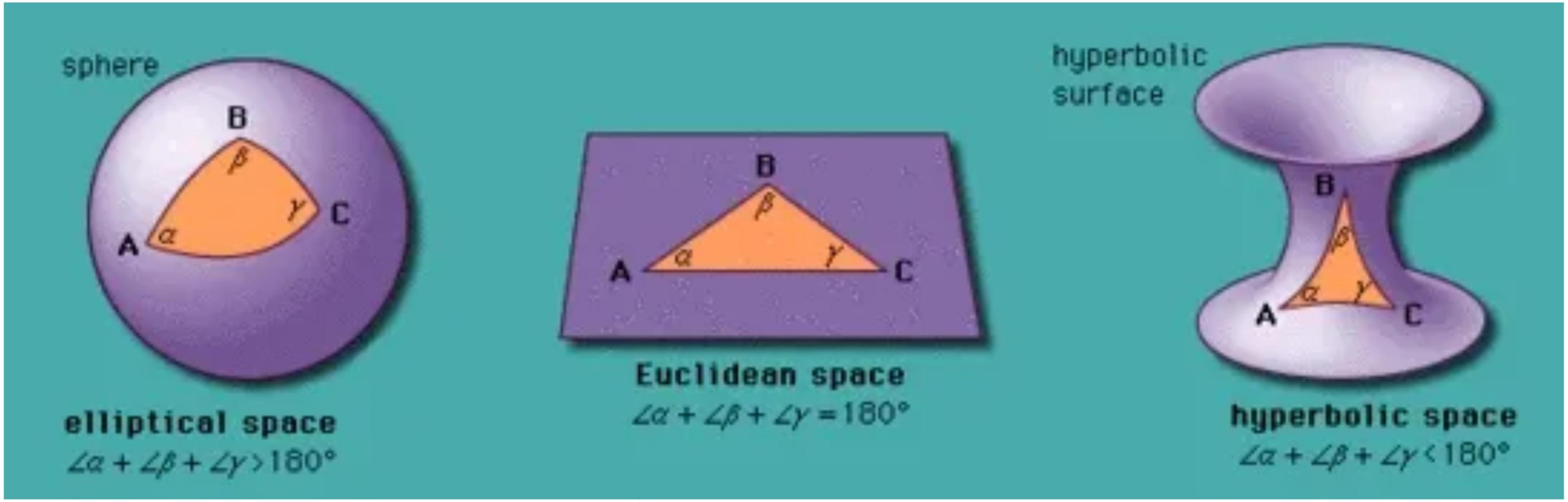

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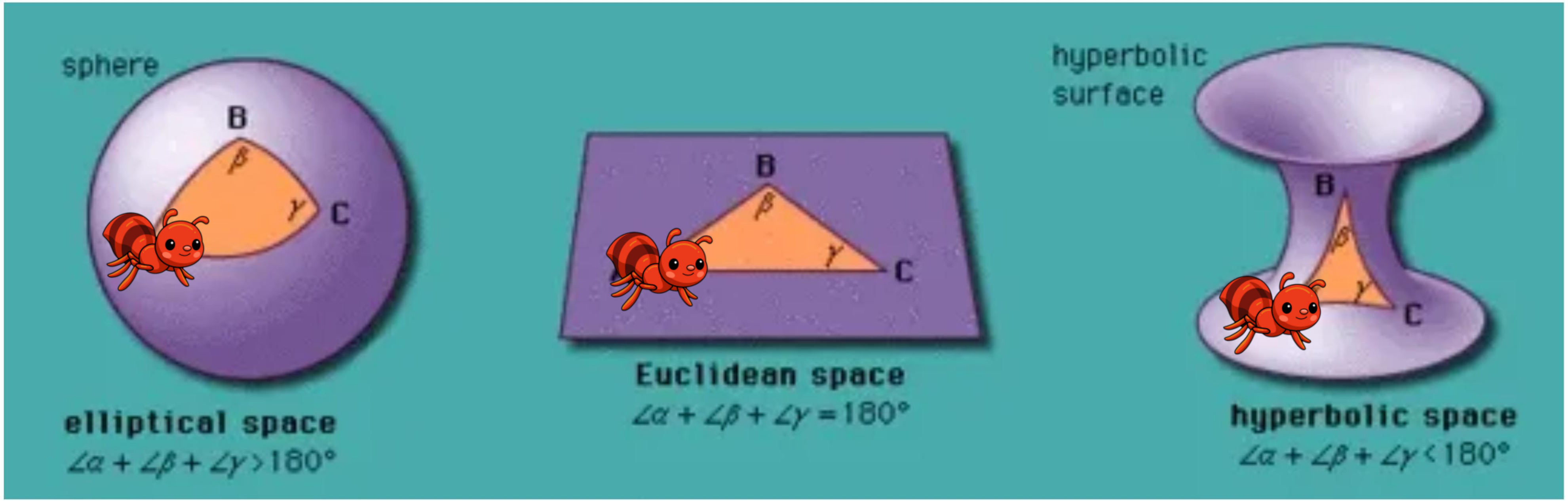

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Intrinsic and extrinsic curvatures

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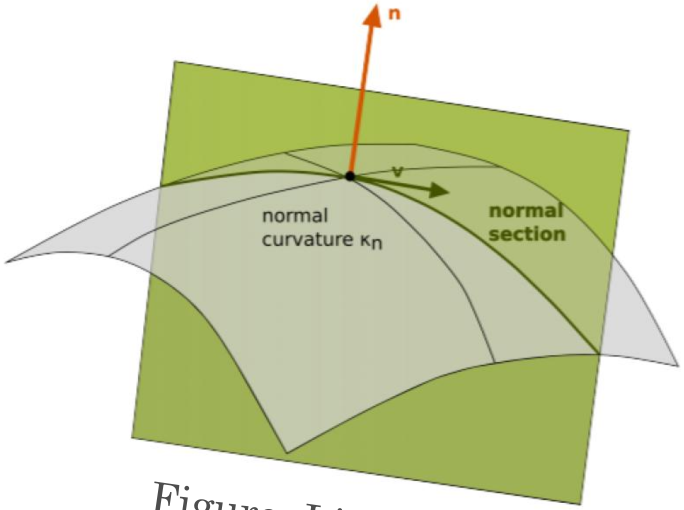

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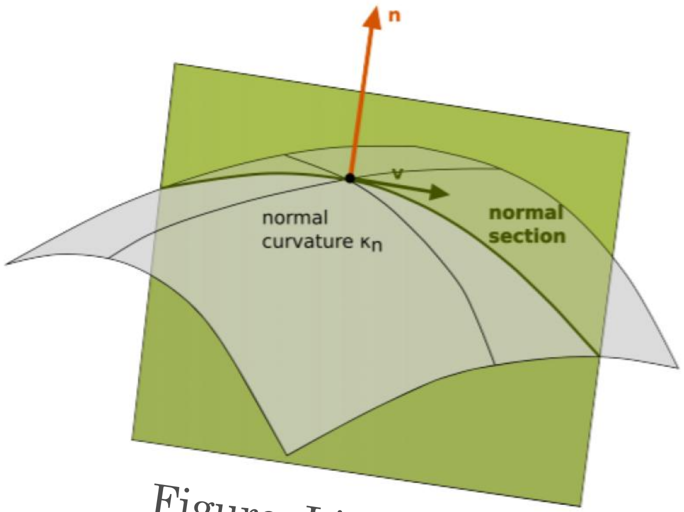

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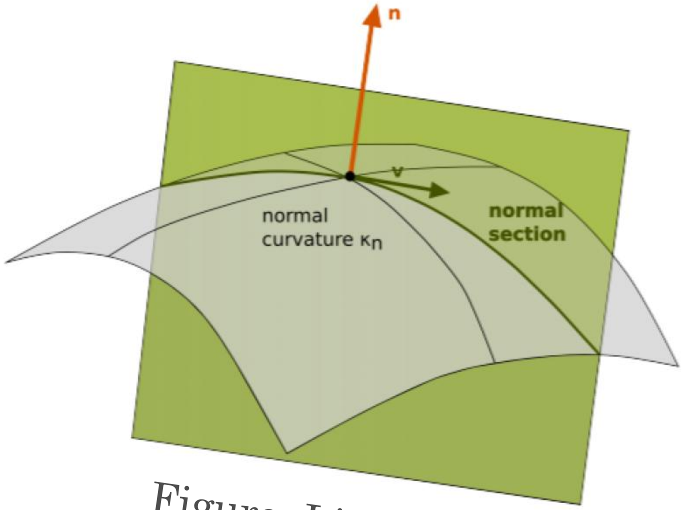

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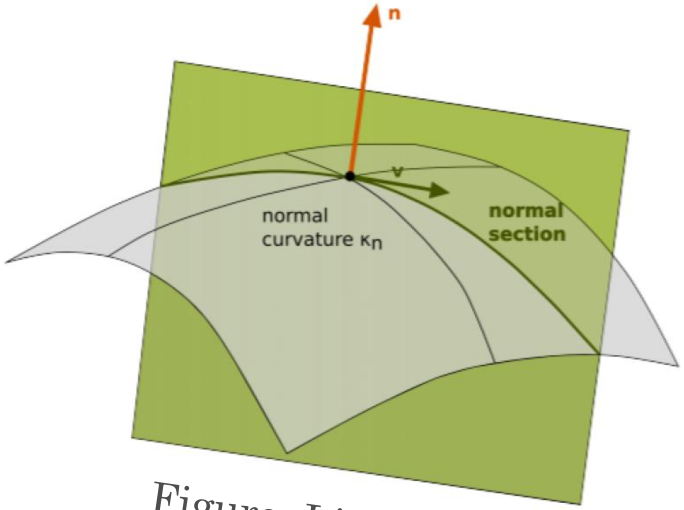

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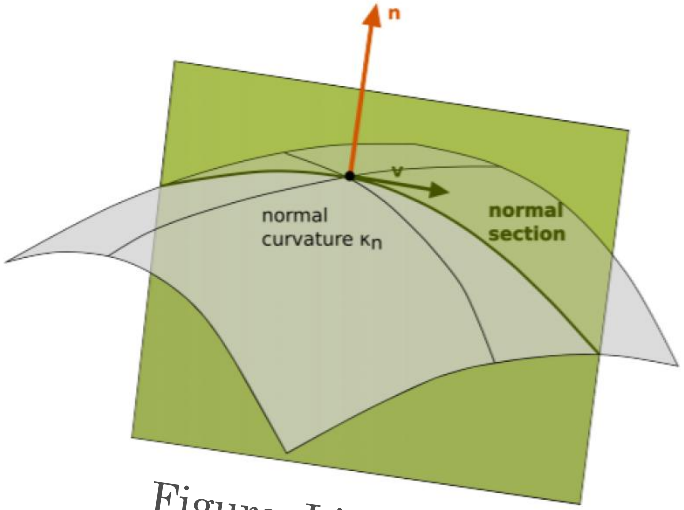

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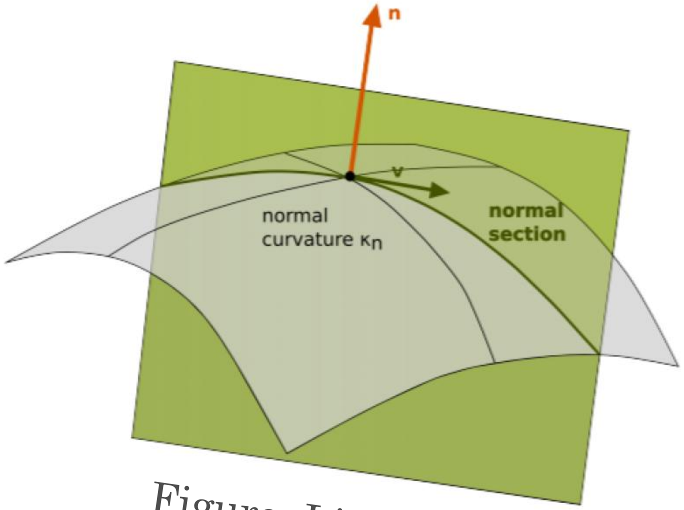

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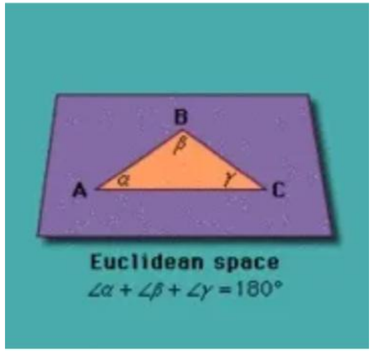
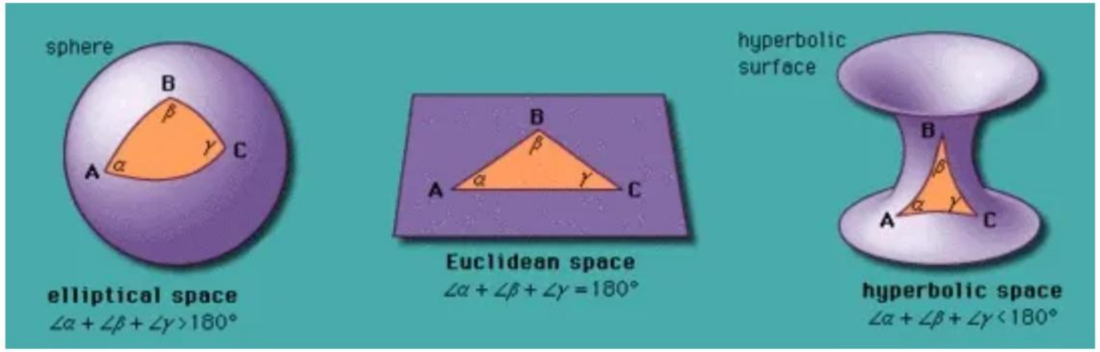


Euclidean Geometry (300 BCE):

1) Line Segment Connection
2) Extension
3) Circle Construction
4) Congruence of Right Angles
5) The Parallel Postulate

Gauss's Theorema Egregium (1827)

Riemannian Geometry (1854)



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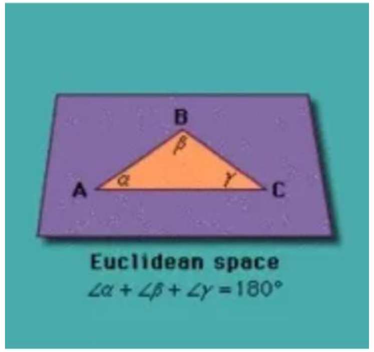

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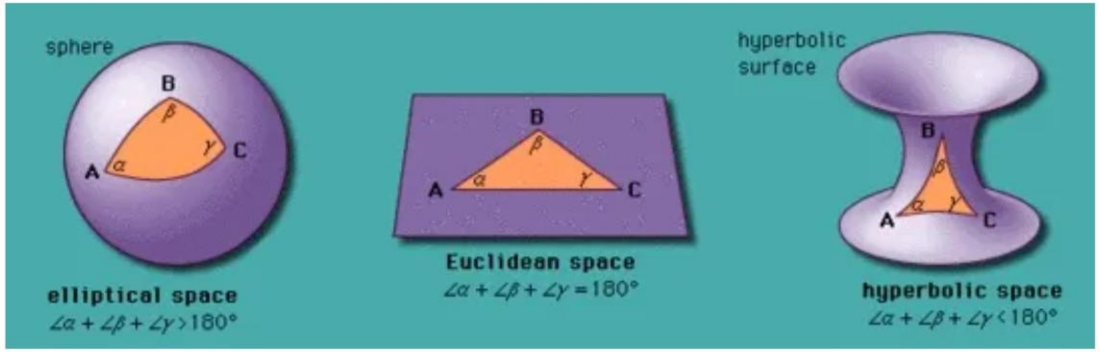

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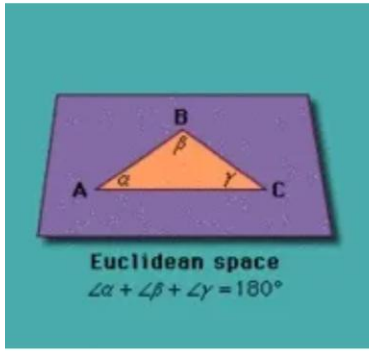

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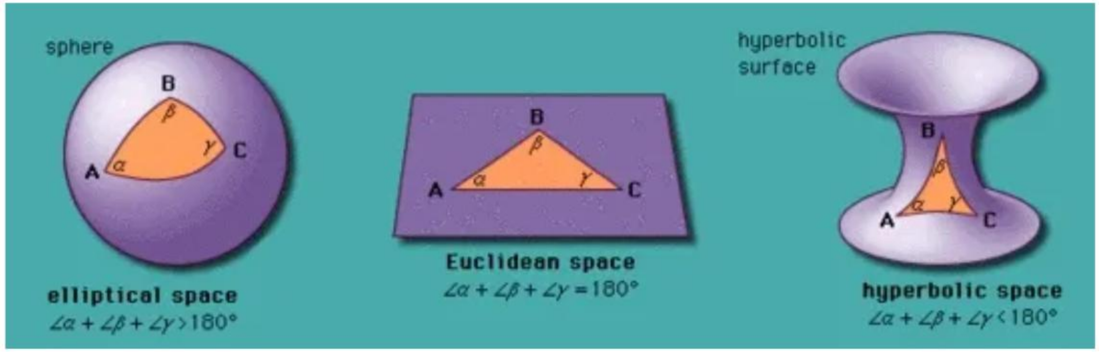

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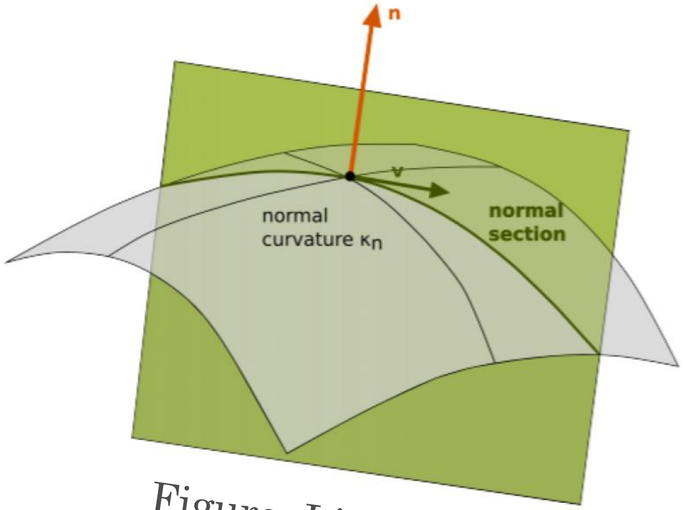

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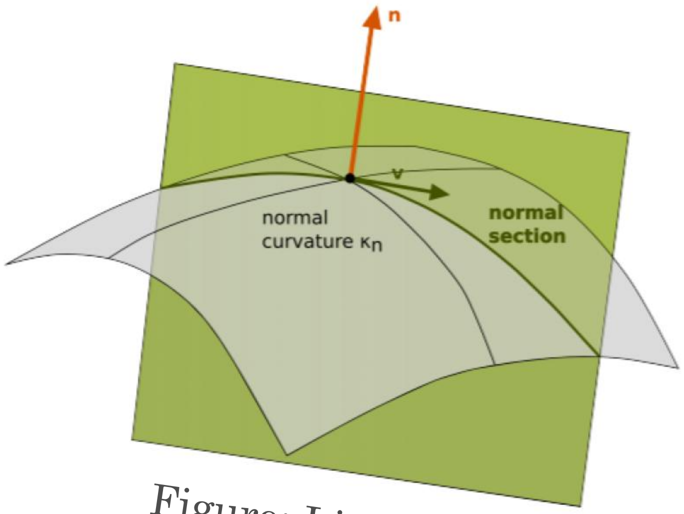

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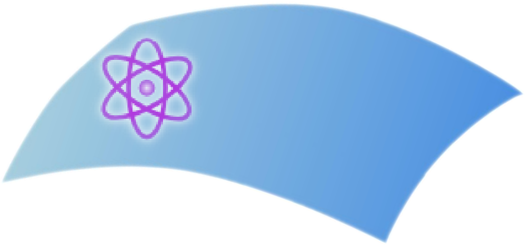

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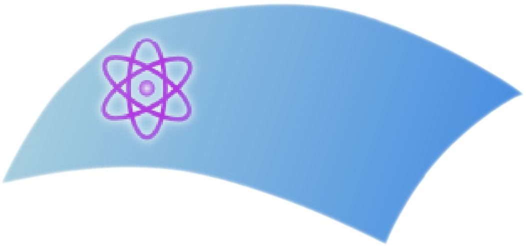

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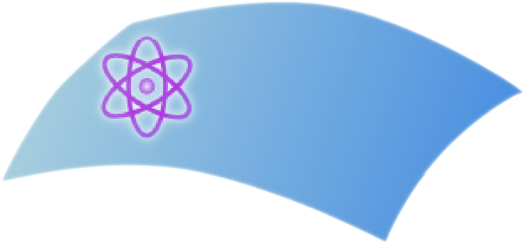

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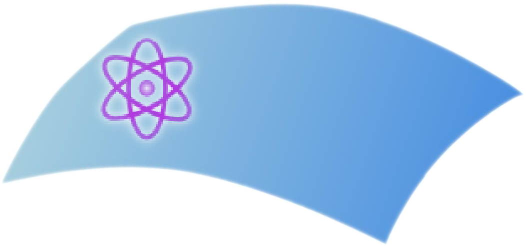

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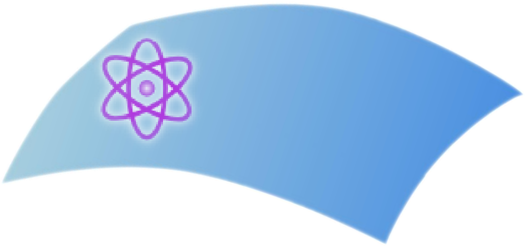

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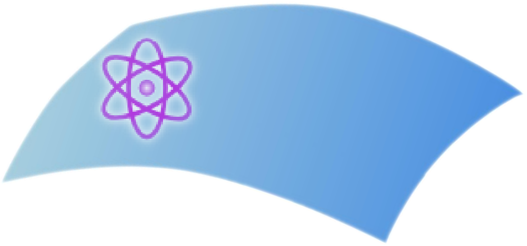

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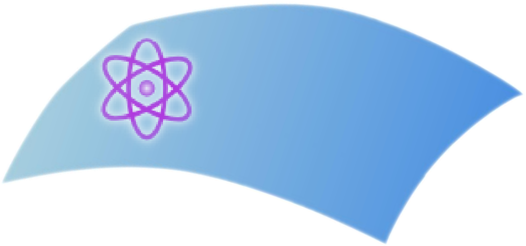

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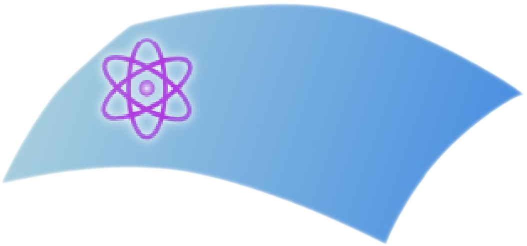

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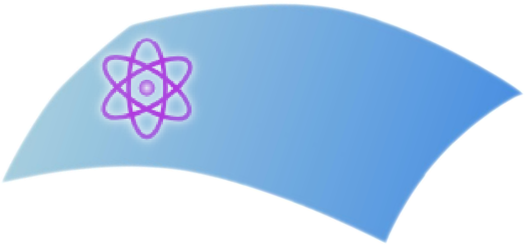

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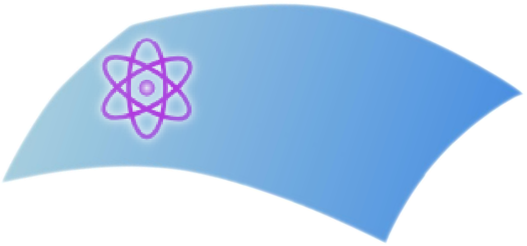

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External potential:

V_{ext} = it depends on the confinement potential

L. Salasnich, A. Parola, L. Reatto, PRA 65, 043614 (2002) Y. Guo, et al, NJP 24, 093040 (2022).

2D Gross-Pitaevskii equation

Geometric potential for a Bose-Einstein condensate on a curved surface
Oliveira and **Móller**, AVS Quantum Science **7**, 033203 (2025)

2D Gross-Pitaevskii equation

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3D Gross-Pitaevskii equation for equilibrium:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar}{2m}\nabla^2\psi + V\psi + g_{int}|\psi|^2\psi$$

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2D Gross-Pitaevskii equation for equilibrium

Geometric potential: $V_{\mathrm{geom}} = -\frac{\hbar^2}{8M}(\kappa_1 - \kappa_2)^2$

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3D Gross-Pitaevskii equation for equilibrium:

$$\mu\psi = -\frac{\hbar}{2m}\nabla^2\psi + V\psi + g_{int}|\psi|^2\psi$$

$$\mu_{2D}\phi = -\frac{\hbar}{2m}\nabla^2_{2D}\phi + \boxed{V_{geom}\phi + V_{ext}\phi} + g_{2D}|\phi|^2\phi$$

2D Gross-Pitaevskii equation for equilibrium

How does the ground state depend on V_{geom} and V_{ext}?

Geometric potential: $V_{\text{geom}} = -\frac{\hbar^2}{8M}(\kappa_1 - \kappa_2)^2$

Thomas-Fermi solution for 2DGP equation

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Thomas-Fermi solution:

$$|\phi|^2 = \frac{\mu_{2D} - V_{geom} - V_{ext}}{g_{2D}}$$

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$$|\phi|^2 = \frac{\mu_{2D} - V_{geom} - V_{ext}}{g_{2D}}$$

Atoms accumulate where potentials sum is lower.

Thomas-Fermi solution for 2DGP equation

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Thomas-Fermi solution for uniform confinement:

$$|\phi|^2 = \frac{\mu_{2D} - V_{geom}}{g_{2D}}$$

Thomas-Fermi solution for 2DGP equation

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2D Gross-Pitaevskii equation for equilibrium

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Thomas-Fermi solution for uniform confinement:

$$|\phi|^2 = \frac{\mu_{2D} - V_{geom}}{g_{2D}} = \frac{1}{g_{2D}}\left(\mu_{2D} + \frac{\hbar^2}{8M}(\kappa_1 - \kappa_2)^2\right)$$

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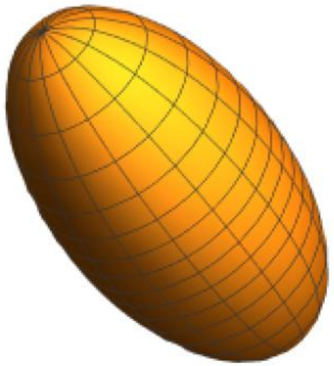
The atoms accumulate where the difference between principal curvatures is higher.

Bubble trap as an ellipsoidal model

Prolate ellipsoid:



Bubble trap as an ellipsoidal model

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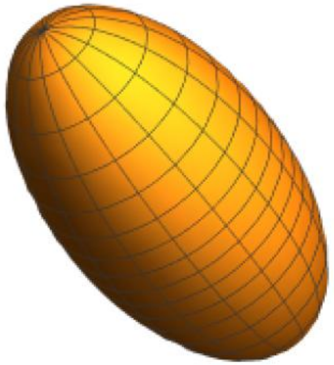

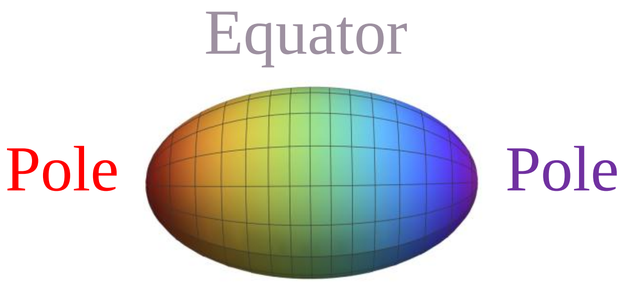

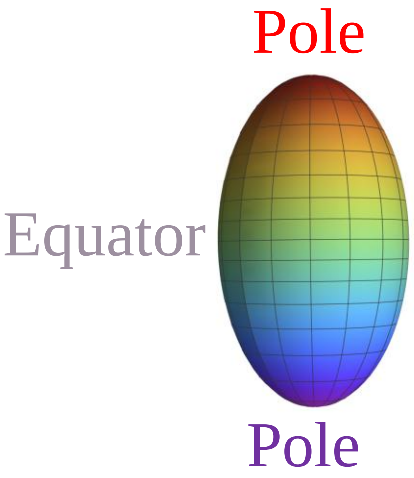

Bubble trap as an ellipsoidal model

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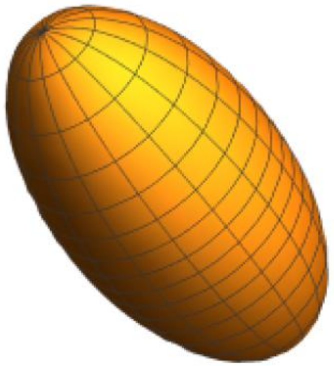

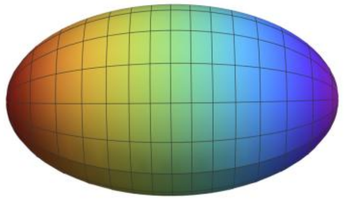

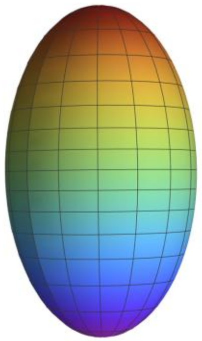

Highest Gaussian curvature on the poles.

Bubble trap as an ellipsoidal model

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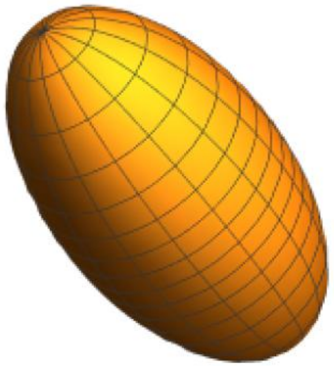
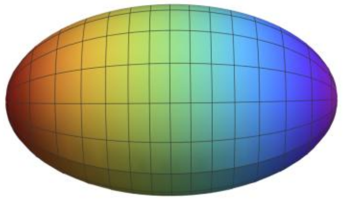
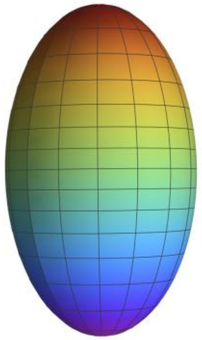


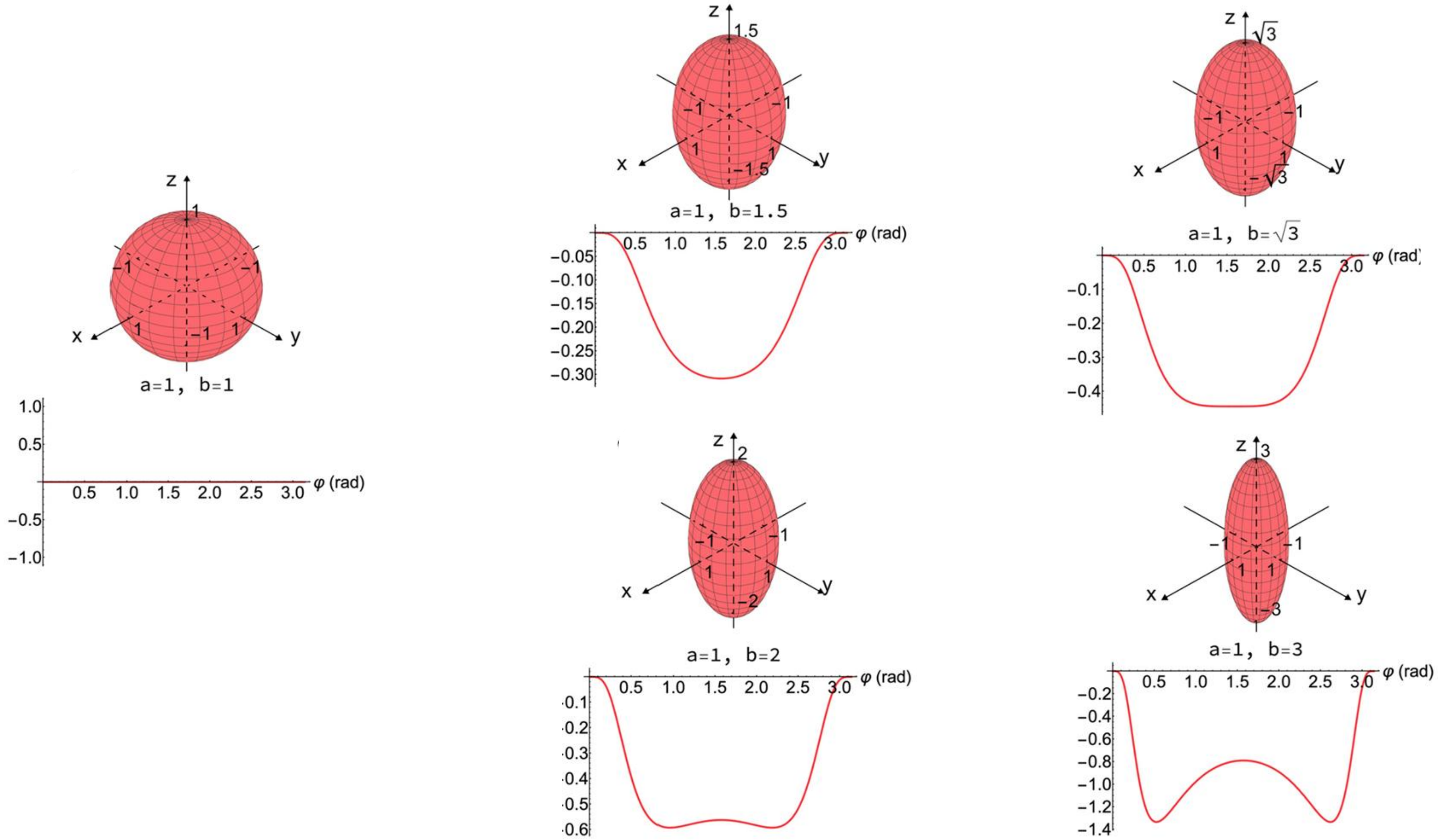




Highest Gaussian curvature on the poles.

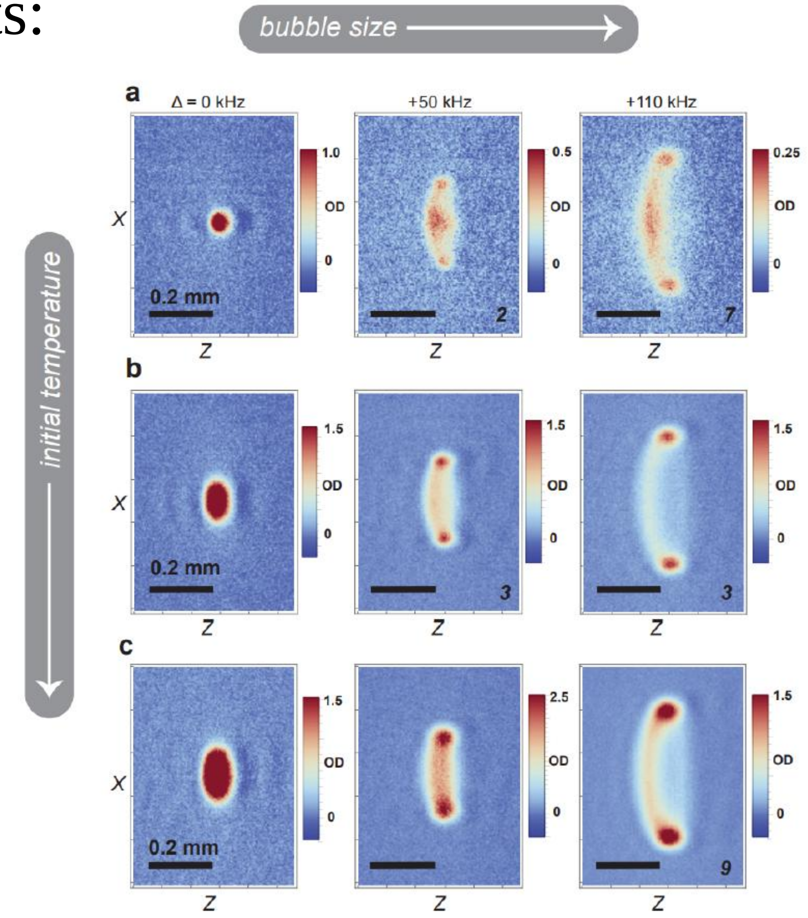
Highest difference between principal curvatures on the equator.

Geometric potential on prolate ellipsoids

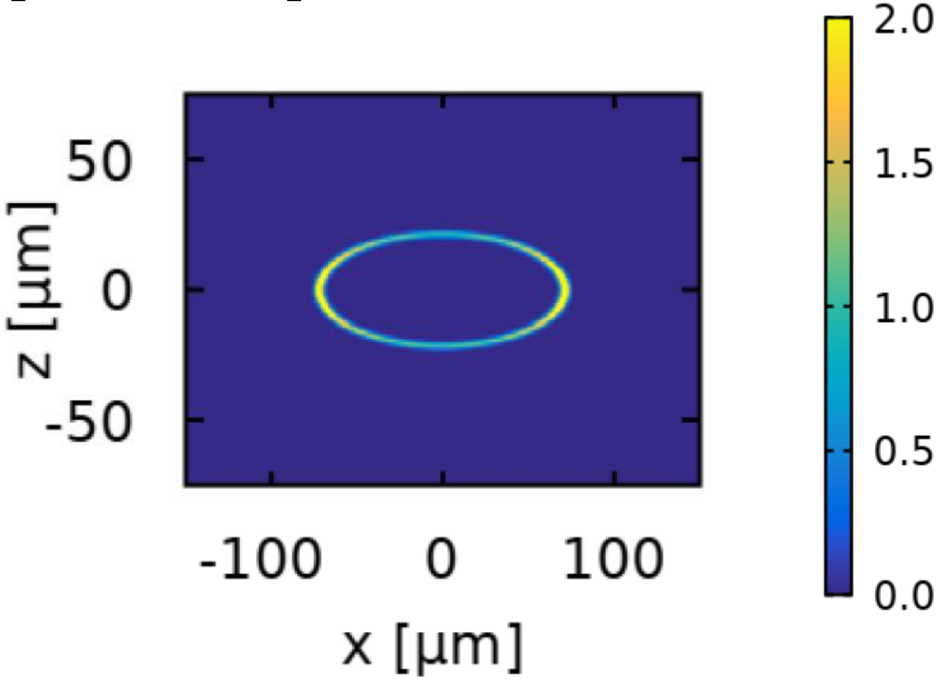


Comparison with literature:

Experiments:



Numerical results on prolate ellipsoid:



R. A. Carollo, D. C. Aveline, B. Rhyno, S. Vishveshwara, C. Lannert, J. D. Murphree, E. R. Elliott, J. R. Williams, R. J. Thompson, N. Lundblad, Nature **606**, 281 (2022)

A. Tononi, F. Cinti, L. Salasnich Phys. Rev. Lett. **125**, 010402 (2020)

Geometric and technical effects

$$|\phi|^2 = \frac{\mu_{2D} - V_{geom} - \beta\, V_{ext}}{g_{2D}}$$

V_{ext} is chosen from the dimensional reduction of the 3D confinement potential of a bubble-trap.

Oliveira and **Móller**, AVS Quantum Science **7**, 033203 (2025)

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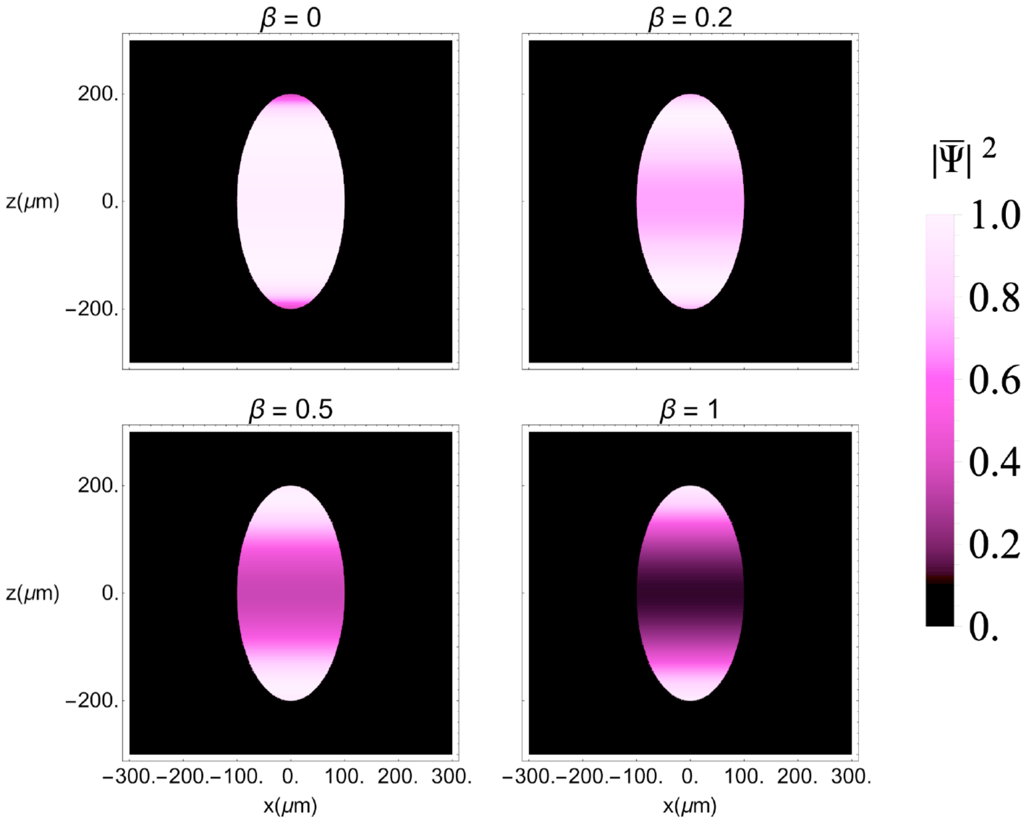

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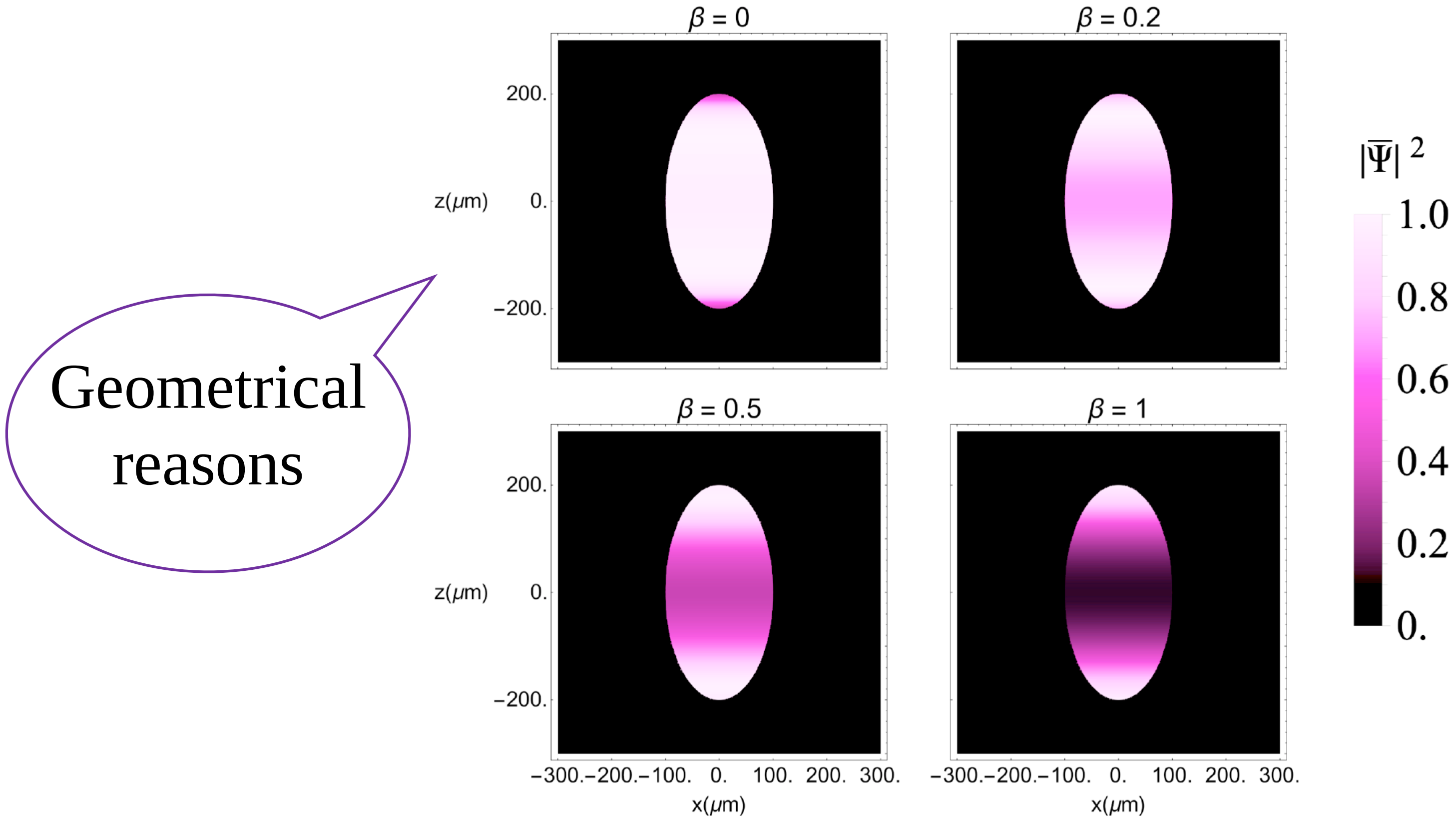

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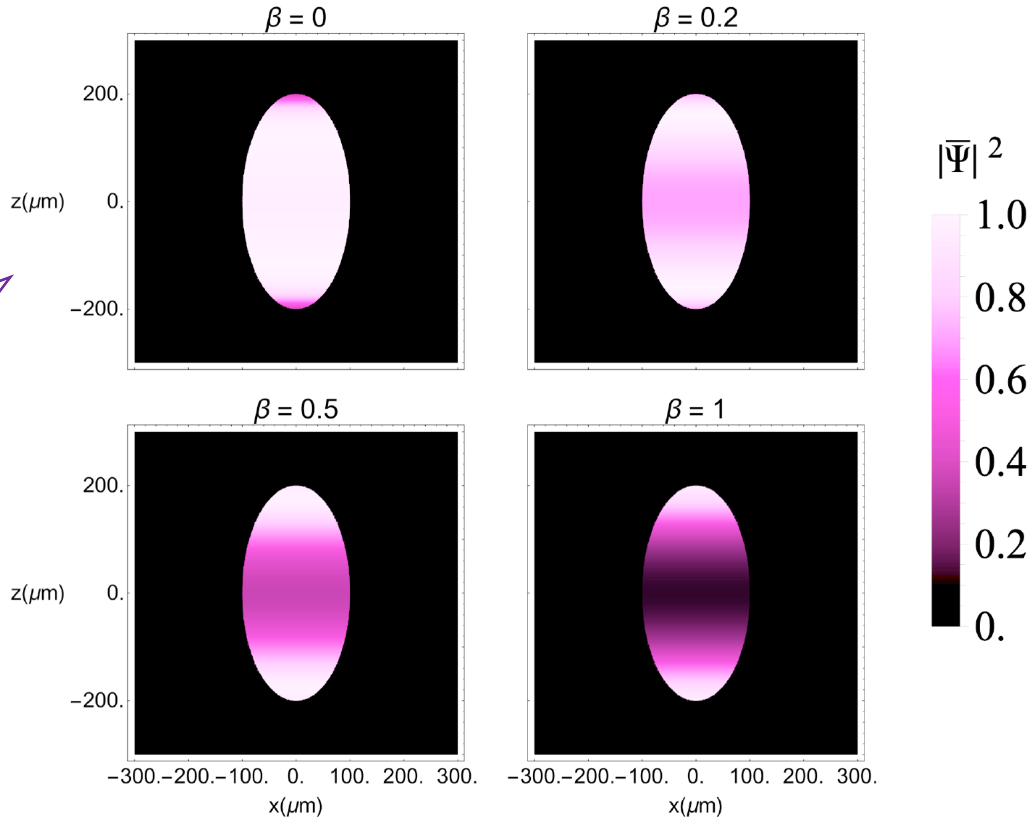

Geometrical reasons

Technical reasons

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Recapitulation

Recapitulation

Bose-Einstein condensation on curved manifolds
Móller, Santos, Bagnato, Pelster
NJP **22**, 063059 (202

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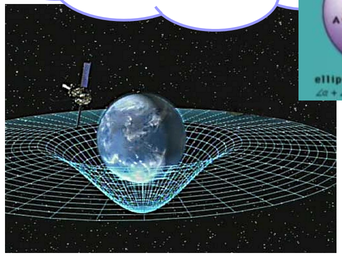
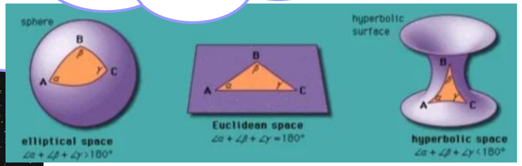
Surface

Manifold





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Surface

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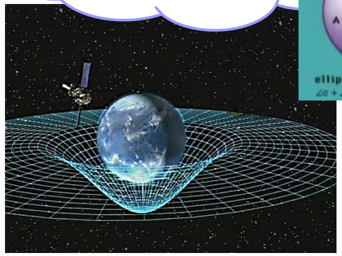

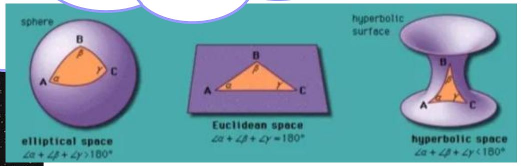

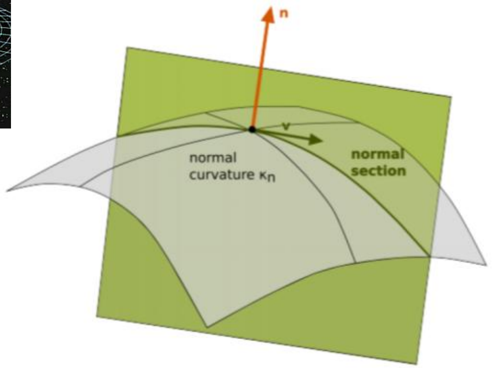

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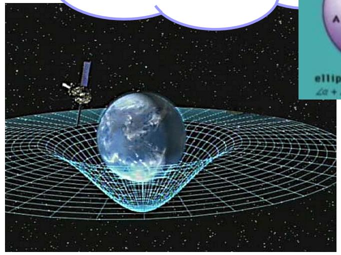
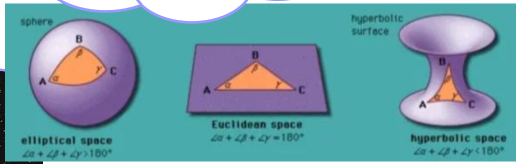
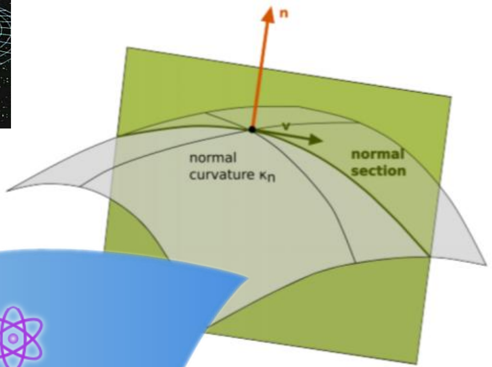
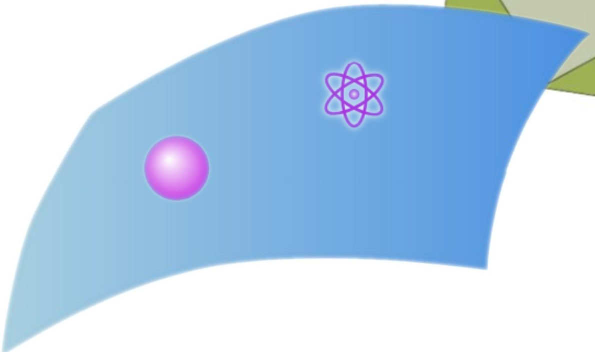
Manifold









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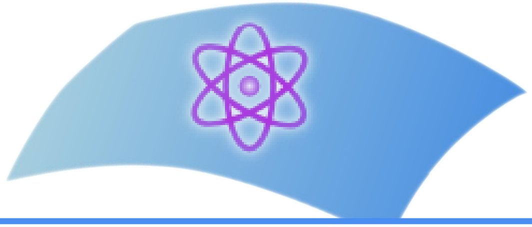

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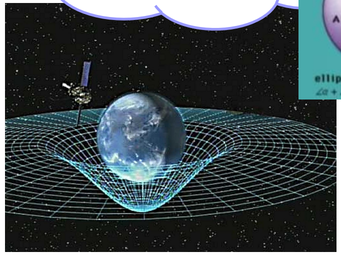
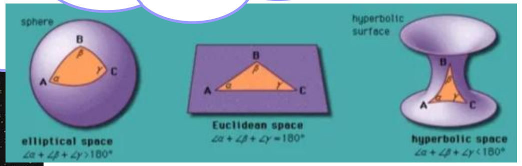
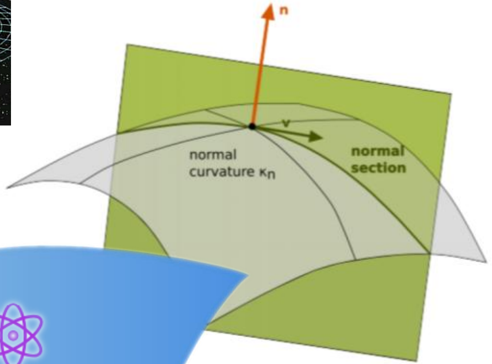
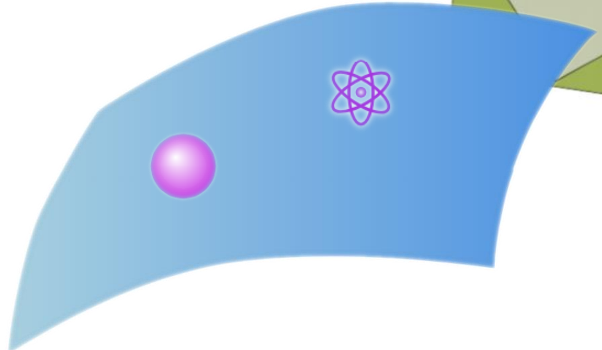
Surface

Manifold









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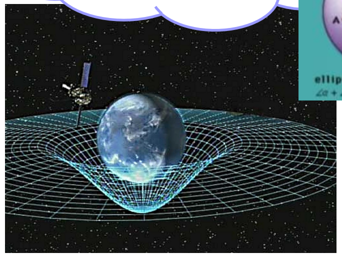
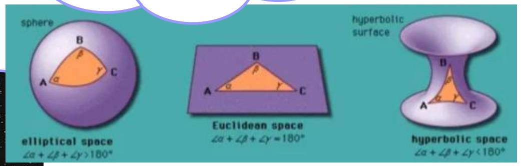
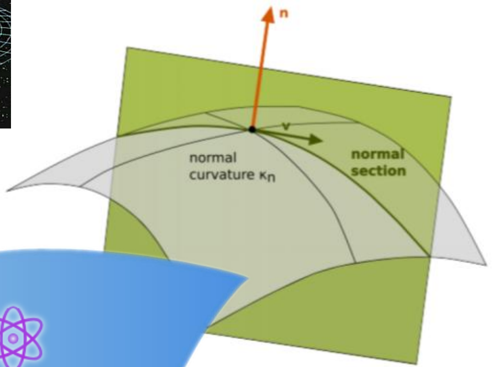
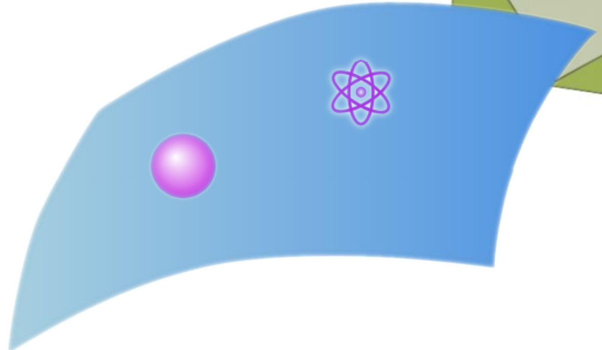
Surface

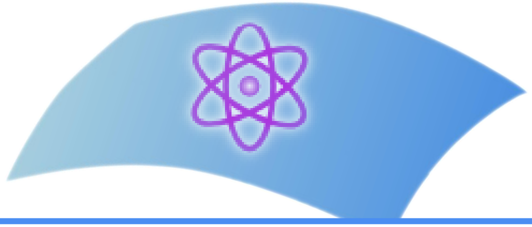
Manifold











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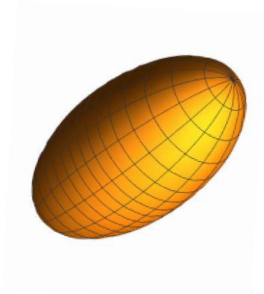

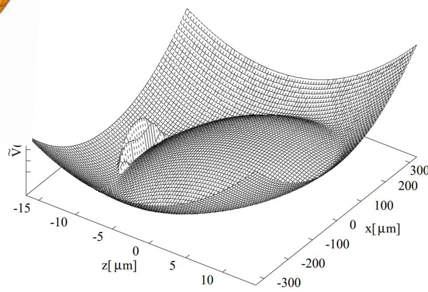

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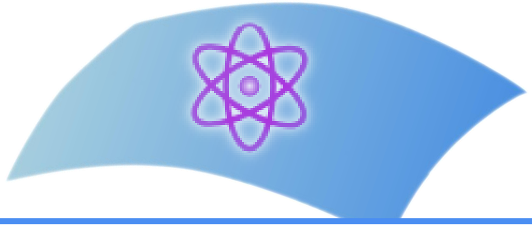

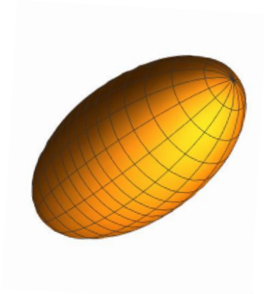

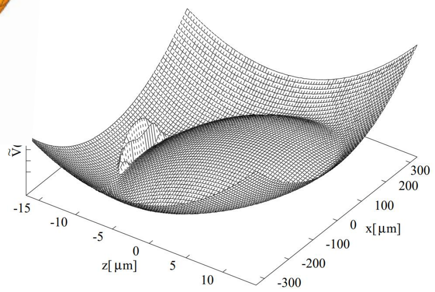

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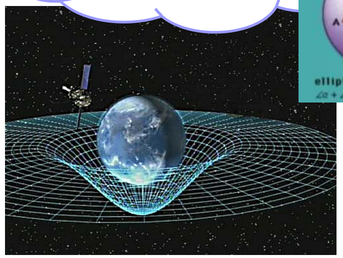
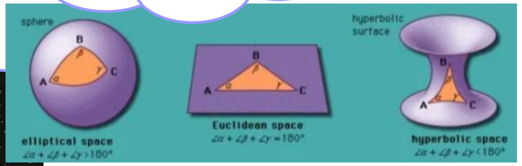
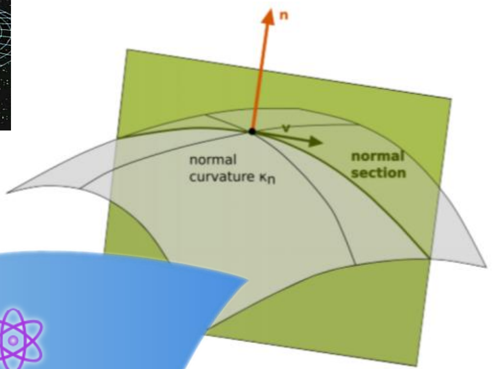
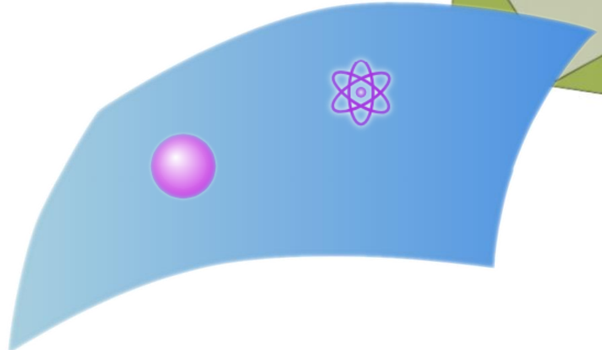
Surface

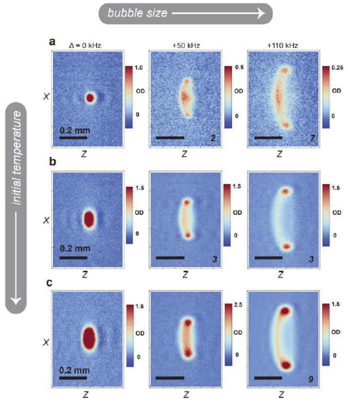
Manifold

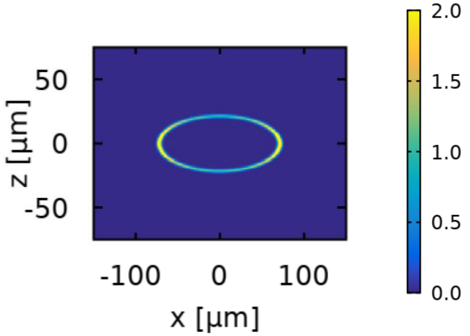












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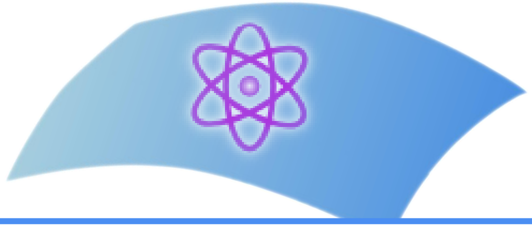
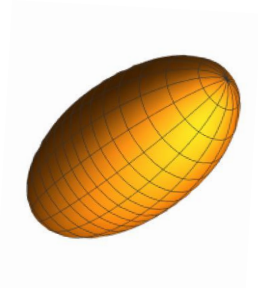
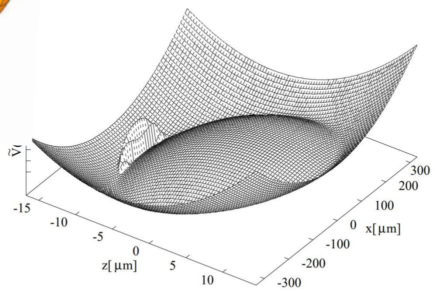
Oliveira and **Móller**, AVS Quantum Science **7**, 033203 (2025)

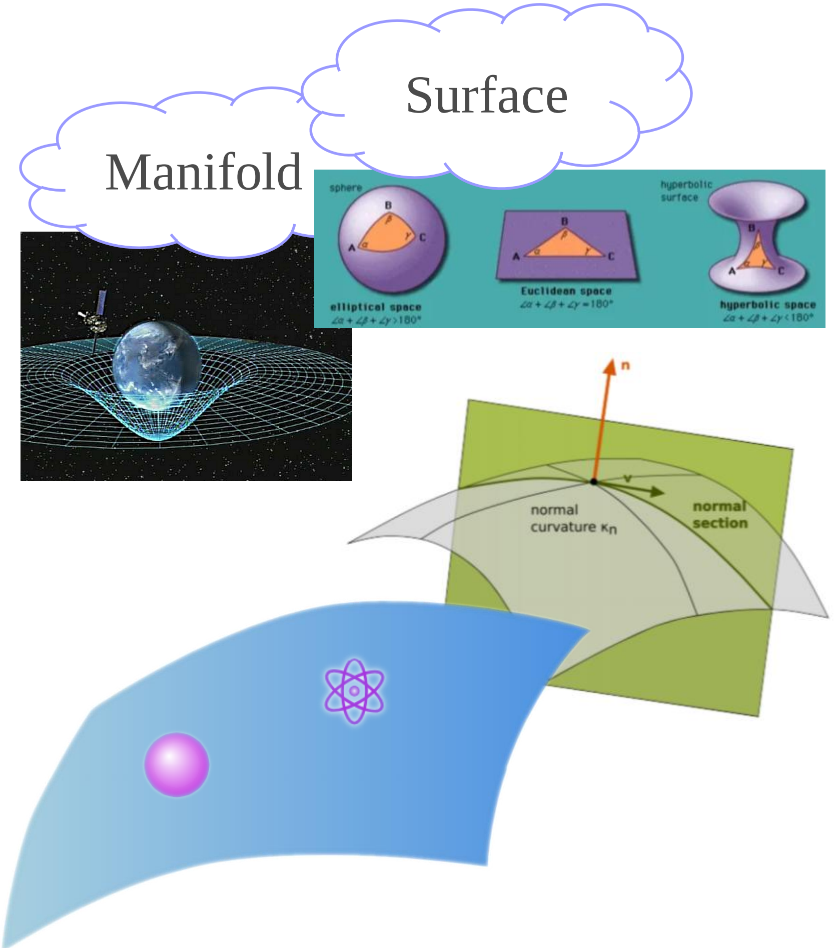


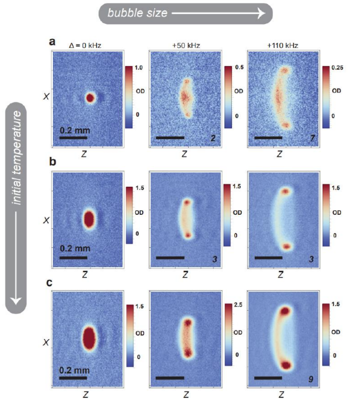


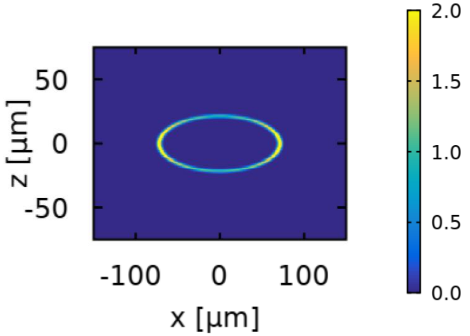


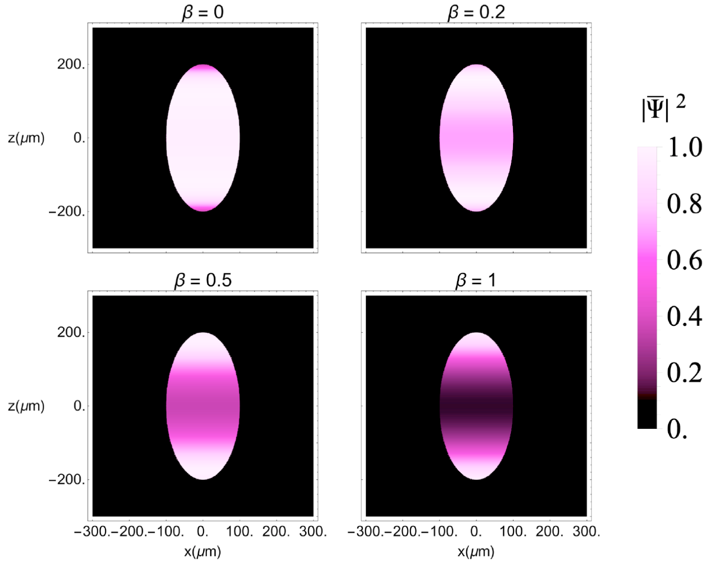
$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar}{2m}\nabla^2\psi + V\psi + g_{int}|\psi|^2\psi$$

$$i\hbar\frac{\partial\phi}{\partial t} = -\frac{\hbar}{2m}\nabla^2_{2D}\phi + V_{geom}\phi + V_{ext}\phi + g_{2D}|\phi|^2\phi$$









Methods

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- We summarize each section at its beginning and ending;
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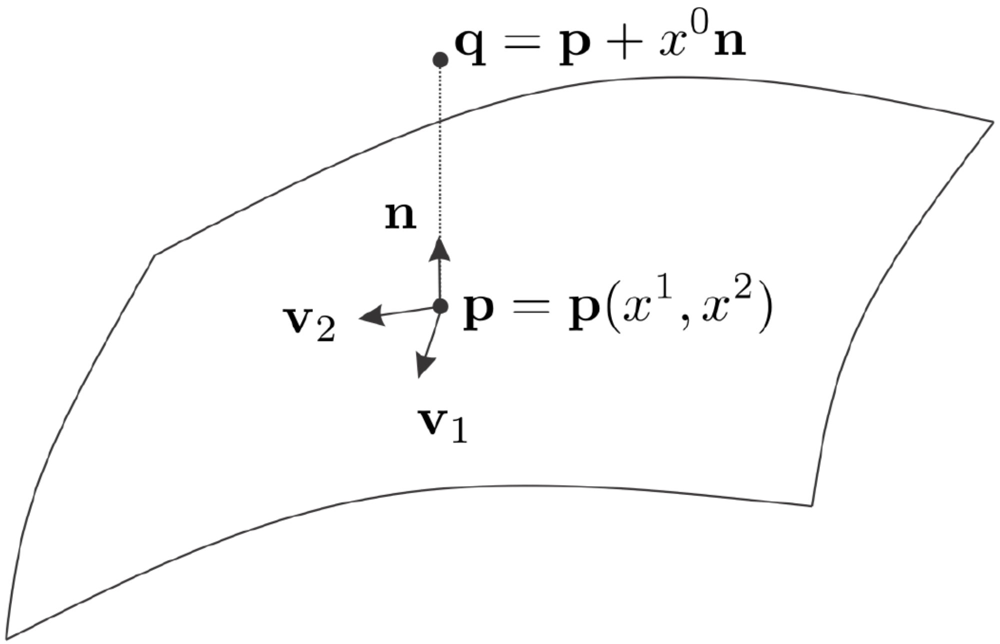
  - Differential and Integral Calculus;
  - Basic notions of Dirac delta.

Model: general smooth surface

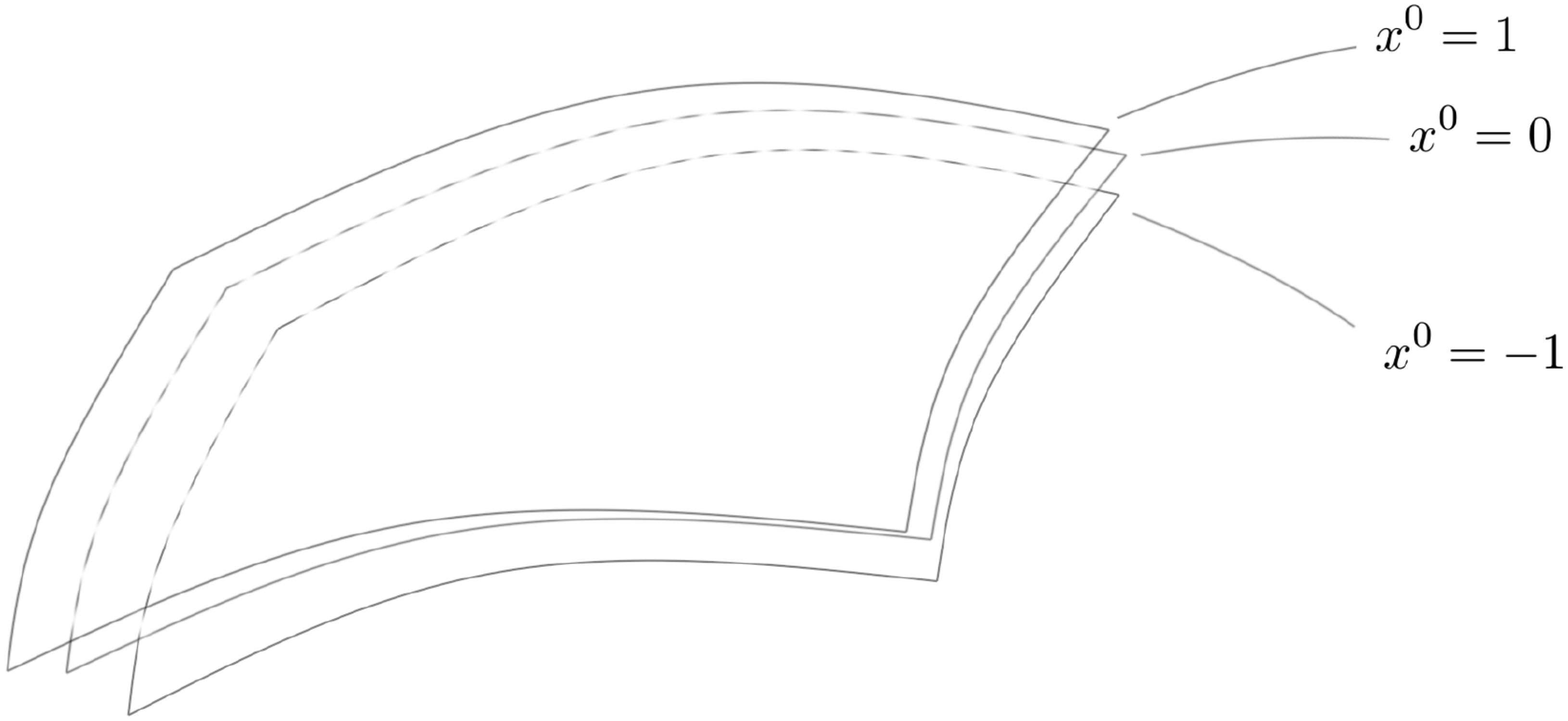
Gaussian normal coordinate system:



Metric: $g(x^1,x^2) = g(x^0 = 0,x^1,x^2)$

Model: general smooth surface

Parallel surfaces:



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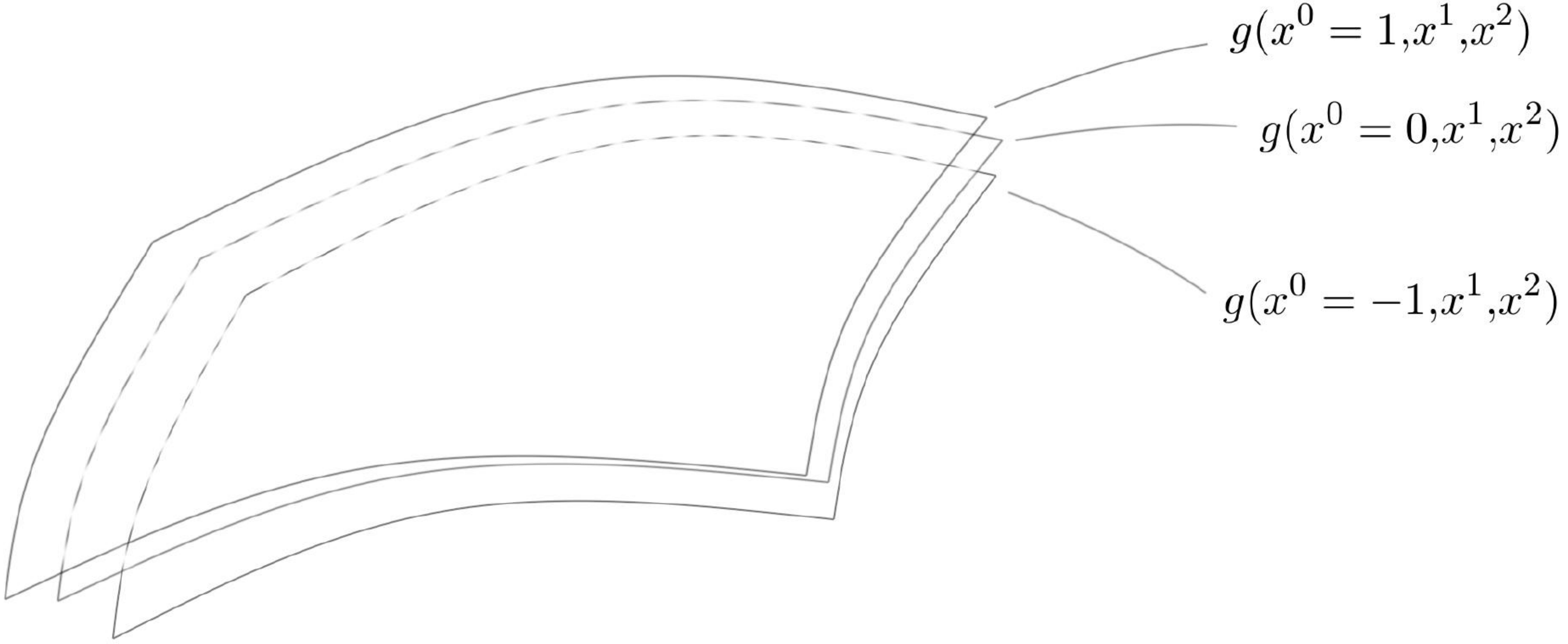

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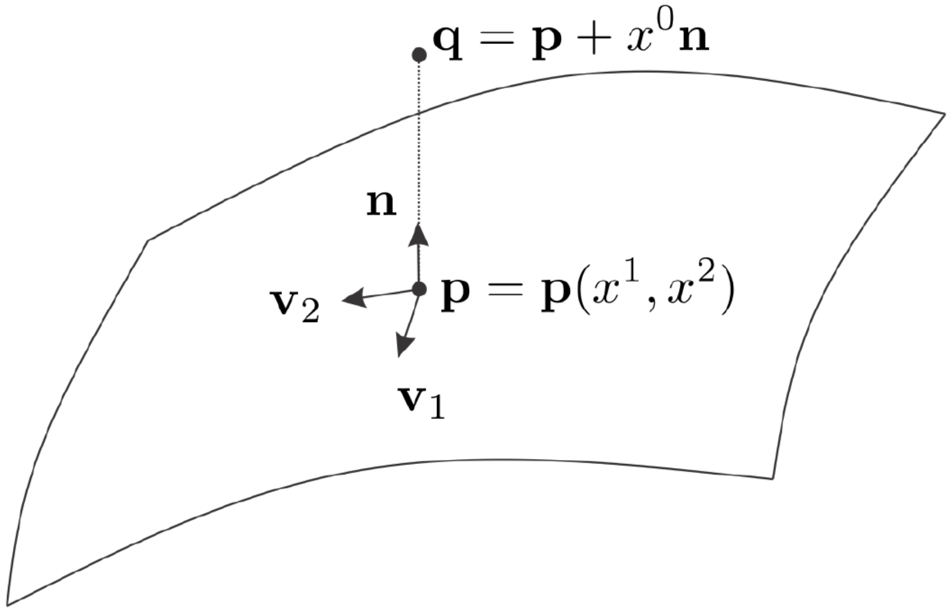

Metric:

$$G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \multicolumn{2}{|c}{\smash{g(x^0, x^1, x^2)}} \\ 0 & & \end{pmatrix}$$

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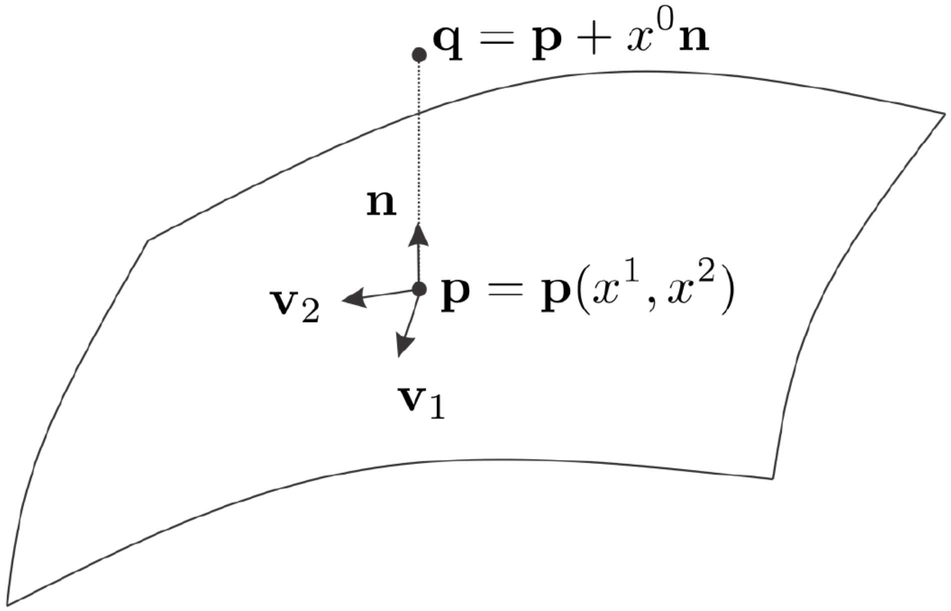

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Metric determinant $\rightarrow$ Jacobian

$$dV = \sqrt{\det G}\, dx^0 dx^1 dx^2$$

Ansatz

Confinement potential:

$$V(x^0, x^1, x^2) = \frac{M[\omega_0 + \omega_2(x^1, x^2)]^2}{2}(x^0)^2$$

$$\mathcal{O}\left(\frac{\omega_0}{\omega_R}\right) = \mathcal{O}\left(\left[\frac{\sigma_0}{R}\right]^{-2}\right)$$

$$\mathcal{O}\left(\frac{\omega_2(x^1, x^2)}{\omega_R}\right) = \mathcal{O}(1)$$

$$\omega_R = \sqrt{\hbar/MR^2}$$

Perturbative expansion

First step: insert the ansatz $\Psi(x^0,x^1,x^2) = \alpha^{1/2} e^{S_0(x^0,x^1,x^2)+S_1(x^0,x^1,x^2)+\dots}$

into the 3DGP $-\mu\Psi - \frac{\hbar^2}{2m}\Delta\Psi + U_{\text{trap}} + g_{\text{int}}|\Psi|^2\Psi = 0$

Inspired by C. Lämmerzahl, A Hamilton operator for quantum optics in gravitational fields, Physics Letters A **203**, 12 (1995)

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We obtain that: $S_0(x^0) = -\frac{(x^0)^2}{2\sigma_0^2}$ $\alpha^{1/2} e^{S_0(x^0,x^1,x^2)} = \mathcal{G}(x^0) = \frac{e^{-(x^0)^2/2\sigma_0^2}}{\sqrt[4]{\pi}\sqrt{\sigma_0}}$

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The ansatz can be rewritten as $\Psi(x^0, x^1, x^2) = \mathcal{G}(x^0)\psi(x^0, x^1, x^2),$

where $\mathcal{G}(x^0) = \dfrac{e^{-(x^0)^2/2\sigma_0^2}}{\sqrt[4]{\pi}\sqrt{\sigma_0}}$ and $\psi(x^0, x^1, x^2) \propto e^{S_1(x^0,x^1,x^2)}.$

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Two-dimensional wave function

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Two-dimensional wave function

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In the third order, the 3DGP becomes:

$$-\mu_2\mathcal{G}\psi - \frac{\hbar^2}{8m}\left(\frac{\kappa_1}{1+\kappa_1 x^0} - \frac{\kappa_2}{1+\kappa_2 x^0}\right)^2\mathcal{G}\psi - \frac{\hbar^2}{2m}(\Delta_{\mathcal{M}(x^0)}\psi)\mathcal{G}$$

$$+\frac{\hbar\omega_2}{\sigma_0^2}(x^0)^2\mathcal{G}\psi + \frac{m\omega_2^2}{2}(x^0)^2\mathcal{G}\psi + g_{\text{int}}\mathcal{G}^3|\psi|^2\psi = 0.$$

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Consider the limit of perfect confinement : $\sigma_0 \to 0$

(Infinitely thin Gaussian.)

Dirac delta

This limit is not well defined:

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Precise definition: Dirac delta

$$\lim_{\sigma_0 \to 0} \int_{\mathcal{N}\{x^0=0\}} f(x^0)\mathcal{G}^2(x^0)dx^0 = \int_{\mathcal{N}\{x^0=0\}} f(x^0)\delta(x^0)dx^0 = f(0)$$

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These integrals are well-defined:

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Dimensional reduction:

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Redefine interactions: $g_{\text{int}} = \sqrt{2\pi}\, g_{2\text{D}}\, \sigma_0$

And obtain the limit: $\lim_{\sigma_0 \to 0} g_{\text{int}} \mathcal{G}^4 = g_{2\text{D}} \delta(x^0).$

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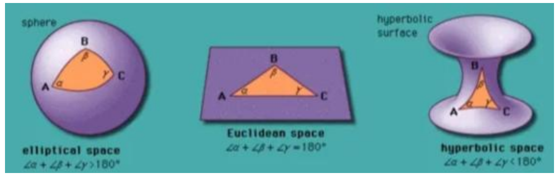
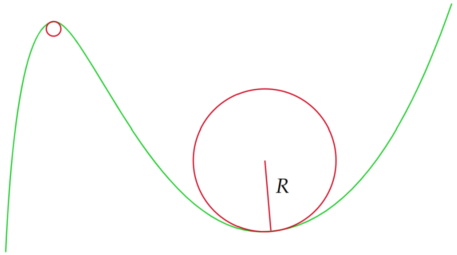
Integrating on x^0, we obtain the 2DGP equation:

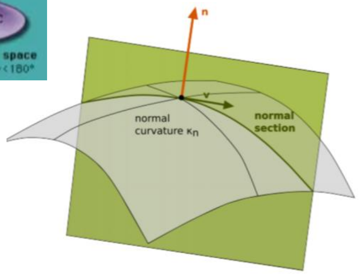
$$-\mu_2\phi - \frac{\hbar^2}{8m}(\kappa_1-\kappa_2)^2\phi - \frac{\hbar^2}{2m}\Delta_{\mathcal{M}}\phi + \frac{\hbar\omega_2}{2}\phi + g_{2\mathrm{D}}|\phi|^2\phi = 0.$$

Summary

We reviewed notions of Geometry:







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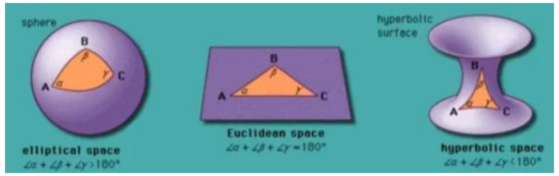
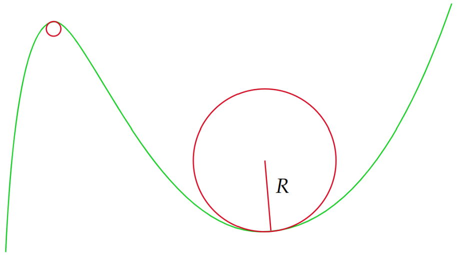
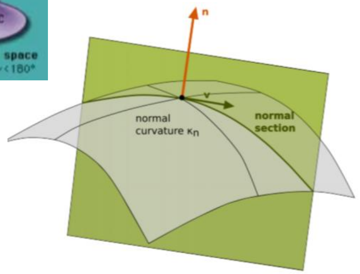
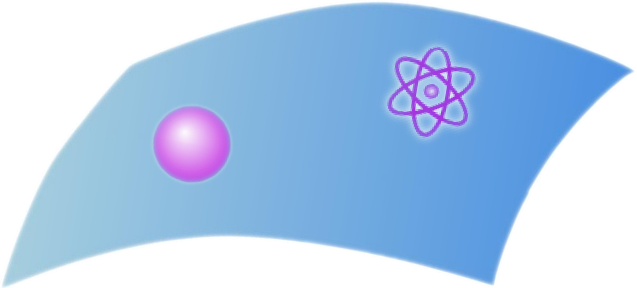






We reviewed physics on curved surfaces:



Summary

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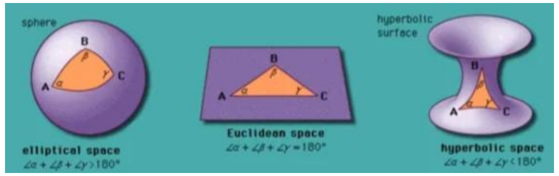

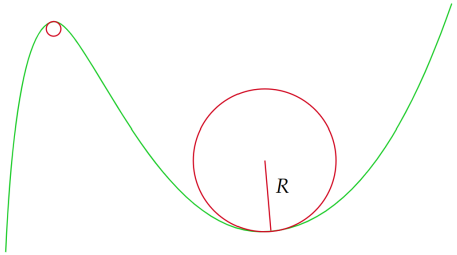

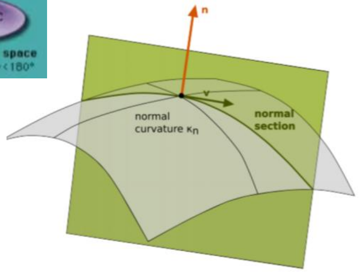

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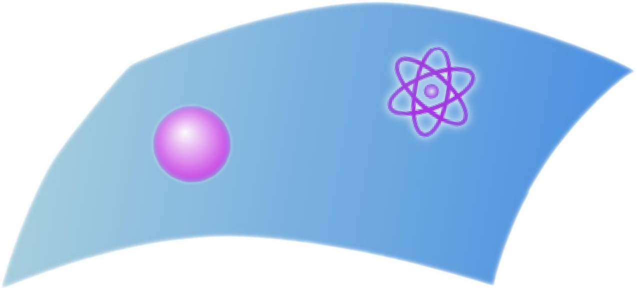

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Oliveira and **Móller**, AVS Quantum Science **7**, 033203 (2025)

Summary

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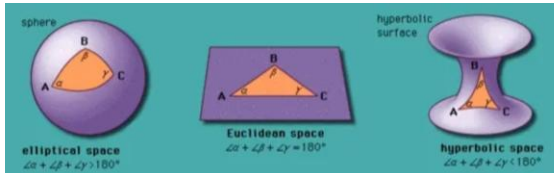

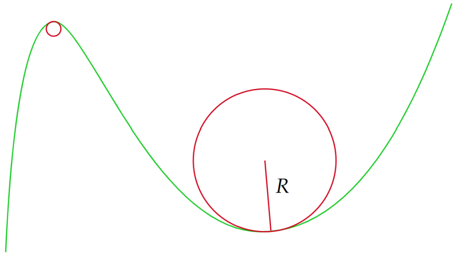

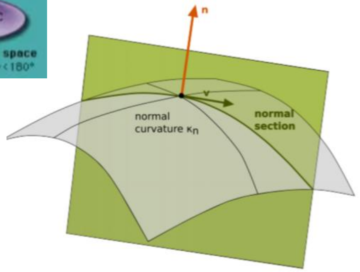

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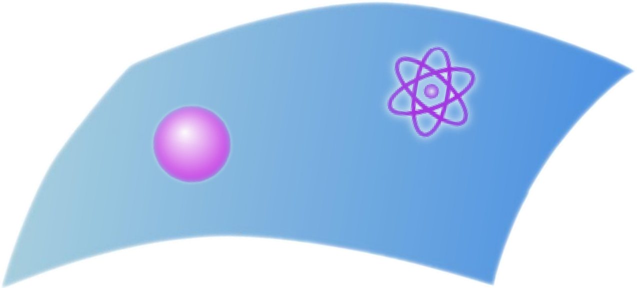
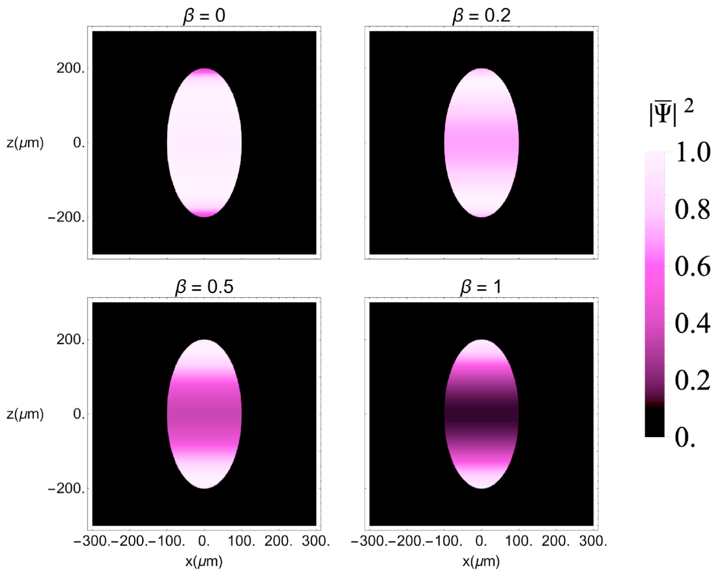


We showed how this affect the understanding of current bubble-trap experiments:



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Oliveira and **Móller**, AVS Quantum Science **7**, 033203 (2025)

Muito obrigada!

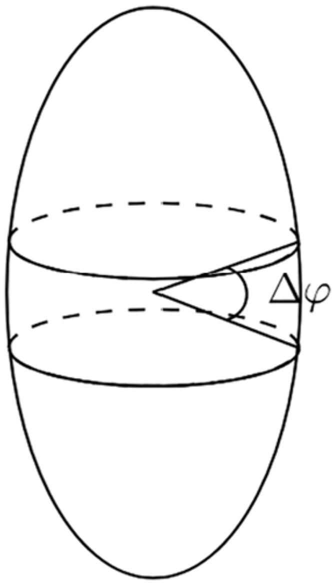
Muchas gracias!

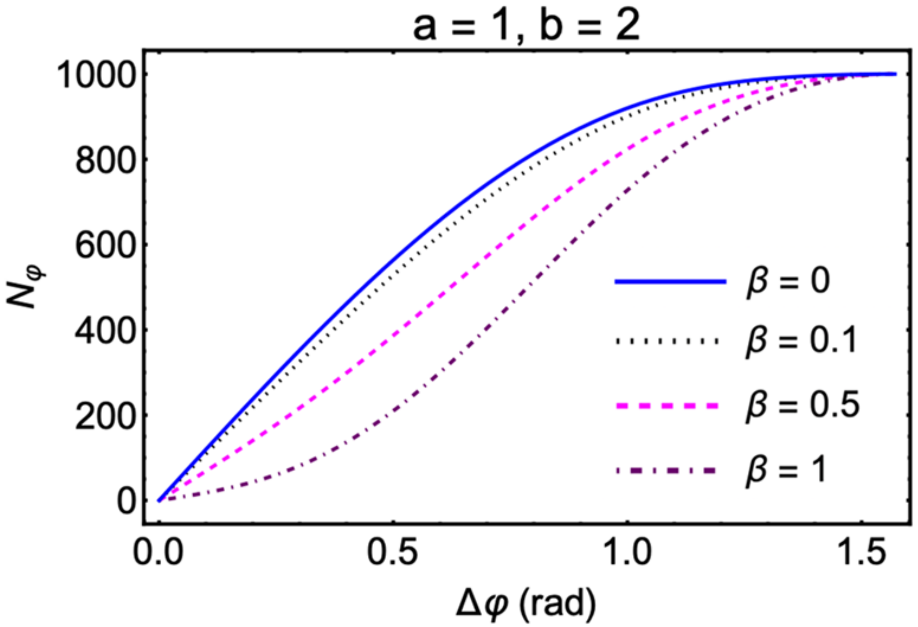
Thank you very much!

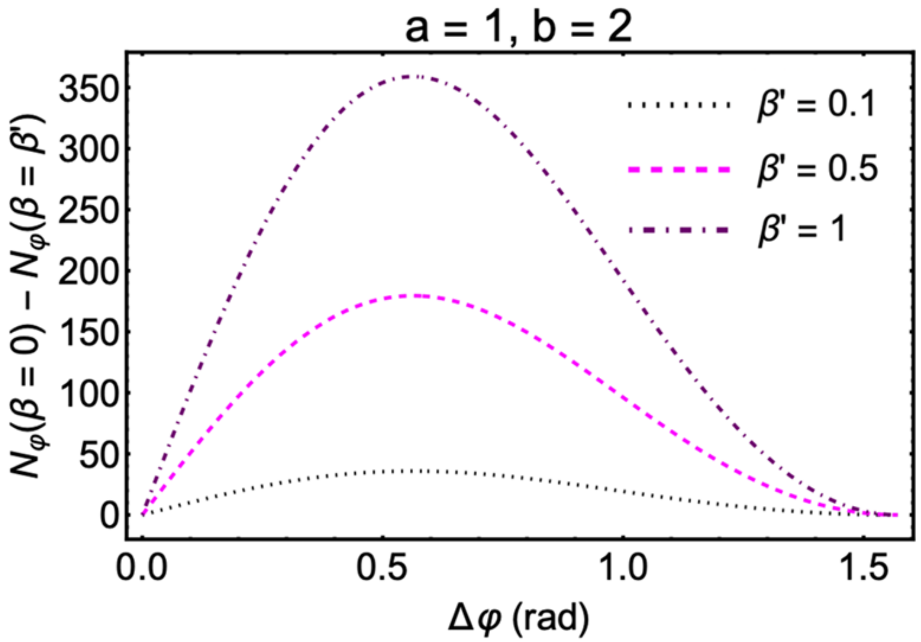
Moltes gràcies!

Ďakujem vel'mi pekne!

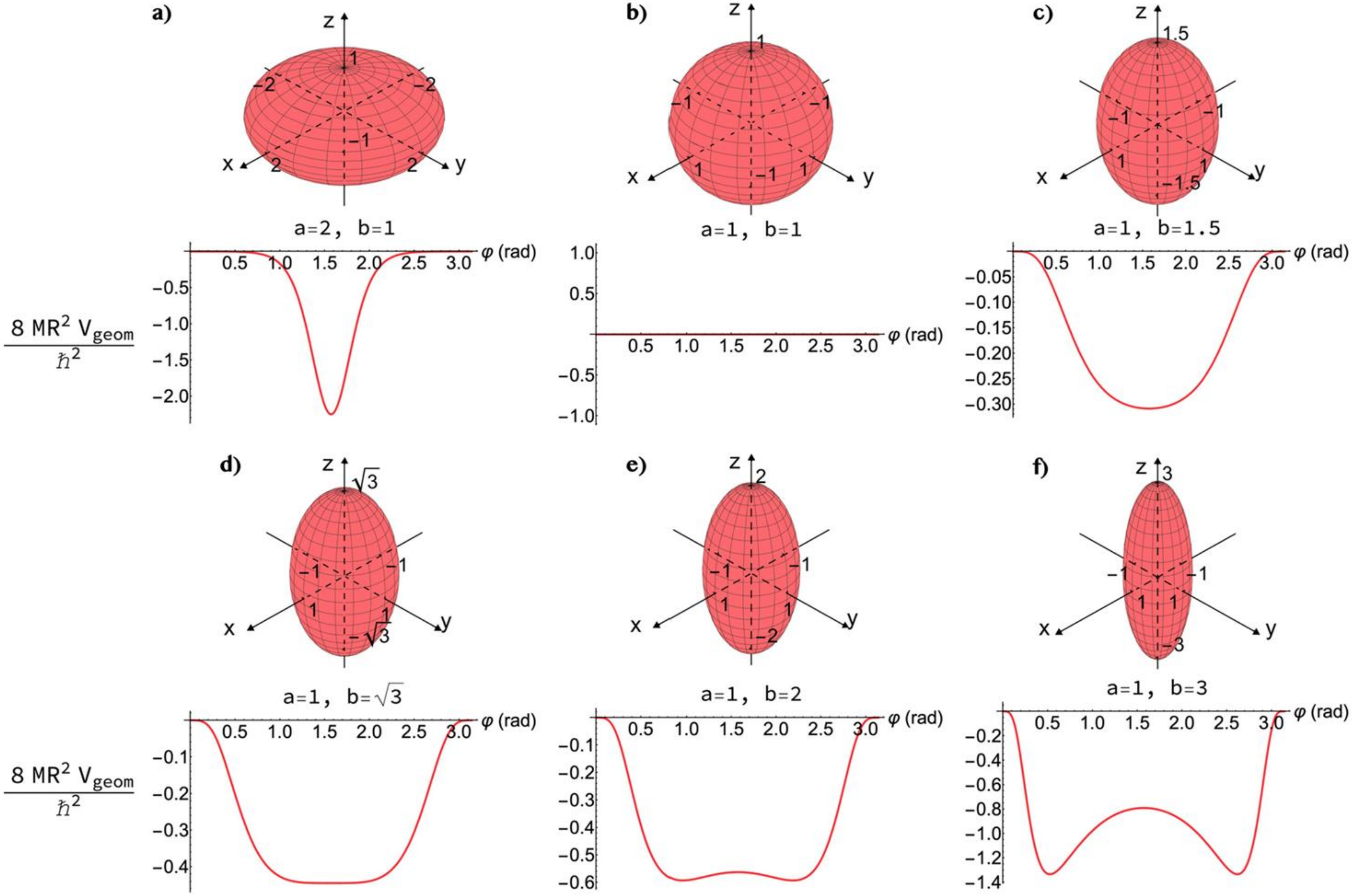
Atom number along the ellipsoid



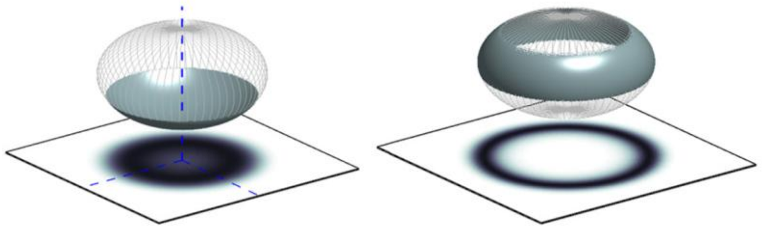


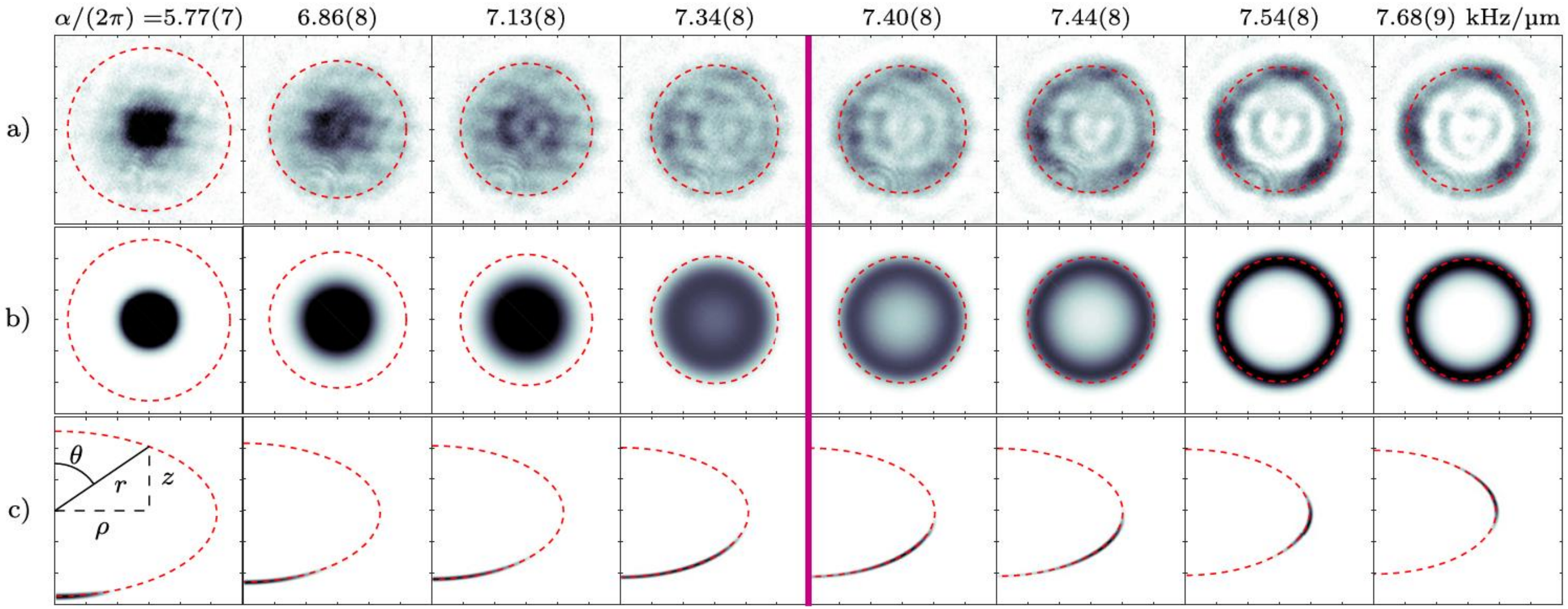


Geometric potential on symmetric ellipsoids



External potential on oblate ellipsoid





Y. Guo, E. M. Gutierrez, D. Rey, T. Badr, A. Perrin, L. Longchambon, V. S. Bagnato, H. Perrin, and R. Dubessy, New J. Phys. 24, 093040 (2022).

Intrinsic and extrinsic curvatures

Sectional curve:

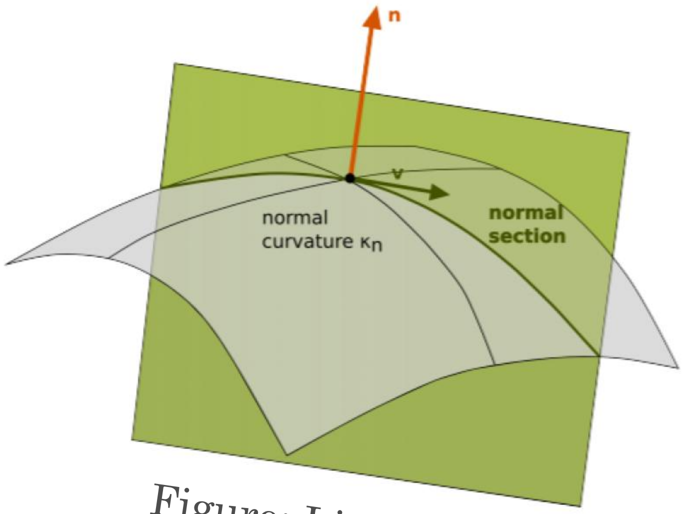


Figure: Lia Vas' notes

Curvature of this curve: κ

Principal (extrinsic) curvatures:

$\kappa_1 = \min \kappa, \qquad \kappa_2 = \max \kappa$

Mean curvature (also "extrinsic"): $H = \dfrac{\kappa_1 + \kappa_2}{2}$

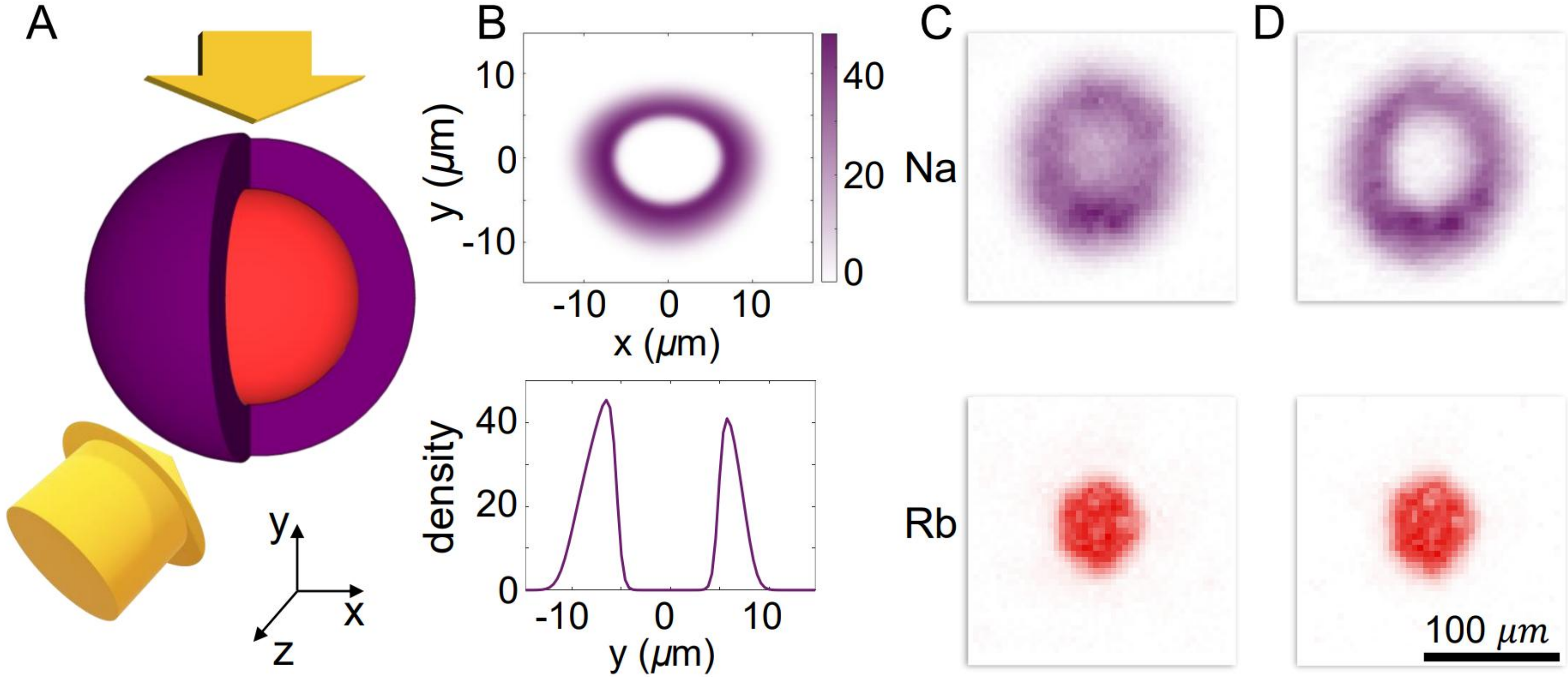
Gaussian (intrinsic) curvature: $K = \kappa_1 \kappa_2$

The Gaussian curvature is an inherent property of the surface.
It is a two-dimensional geometric property.

We still need the normal vectors to compute the principal curvatures and mean curvature.
They are three-dimensional geometric properties.

BEC on a bubble trap with dual species

Recent experimental attempts using dual species (2022):



F. Jia, Z. Huang, L. Qiu, R. Zhou, Y. Yan, D. Wang, Expansion dynamics of a shell-shaped Bose-Einstein condensate, Phys. Rev. Lett. **129**, 243402 (2022)

Overview

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- Consider the Gaussian normal coordinate system, parallel surfaces and metric;
- Consider confinement potential and appropriate limits;
- Consider general ansatz and perform perturbative expansion: $e^{S_0(x^0,x^1,x^2)+S_1(x^0,x^1,x^2)+\dots}$
- Fisrt order: we find that e^{S_0} is a square root of Gaussian: $\mathcal{G}(x^0)$
- Second order: we find a general form for $e^{S_1} = \psi$:

$$\frac{\mathcal{G}(x^0)\phi(x^1,x^2)}{\sqrt{(1+\kappa_1 x^0)(1+\kappa_2 x^0)}} \rightarrow \psi(x^0,x^1,x^2)$$

- Third order: we obtain 3D equation for ψ.
- To find a 2D equation for ϕ we must deal with Dirac deltas appropriately.
- Finally, we integrate the 3DGP on the x^0 variable, yielding the 2DGP.