Superfluid Vortex Dynamics on Curved Surfaces

Workshop and School on "Frontiers in Ultracold Quantum Gases"
XLI Trobades Científiques de la Mediterrània – Josep Miquel Vidal

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Overview

1. Introduction
   - Bose-Einstein condensate (BEC) of ultracold gases
   - "Bubble" BECs
2. Superfluid velocity field
   - Stream Function
   - Vortex in the complex plane
   - Vortex dynamics
3. Superfluid on a sphere
   - Stereographic projection
   - Dipole potential in the Sphere
   - Dynamics of a Vortex dipole
   - Four vortices configurations
4. Vortex Dynamics on General Compact Surfaces
5. Conclusions

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Bose-Einstein condensate of ultracold gases

The remarkable creation (1995) of Bose-Einstein condensates (BECs) in ultracold dilute atomic gases - platform to simulate condensed matter models - explore exotic quantum phases in totally new regime of parameters.

Quantum gases benefit from a high degree of control of their external and internal degrees of freedom by electromagnetic fields.

• Feshbach Resonances: tune the atomic scattering length (interaction strength).
• Lasers to produce stirring and deformed traps: BEC excitations (vortices and collective modes).
• Confinement geometry: bulk condensates ranging from flat pancakes to elongated cigars, and various periodic potentials (optical lattices).
- low dimensional systems (effective 1D, 2D systems).
- topology: ring potential, "Bubble trap".
• Easy optical detection (fluorescence and optical absorption to obtain atomic number and density).

$\Rightarrow$ An ideal system for the study of superfluid dynamics
- the vortex dynamics on curved surfaces.

"Bubble" BECs

Proposals to create thin shell traps leading to "bubble" BECs:

• Adiabatic dressed potentials[1,2]- Magnetic field gradients can be used to compensate gravity[3]

• NASA Cold Atom Lab - Bubble-trapped BECs have in microgravity environments[4,5].

• Shell-shaped condensate using a repulsive dual-species mixture[6].

$\Rightarrow$ One of the aims of these bubble-trap experiments is the generation of vortices by rotating the dressed trap or through the spontaneous creation of vortex–antivortex pairs across the condensation transition.

• Influence of inhomogeneities in realistic trapping potentials.

1. Y. Colombe et al., Europhys. Lett. 67, 593 (2004).
2. B. M. Garraway and H. Perrin, J. Phys. B 49, 172001 (2016).
3. Y. Guo et al., New J. Phys. 24, 093040 (2022).
4. D. C. Aveline et al., Nature 582, 193 (2020).
5. R. Carollo et al., Nature 606, 281 (2022).
6. F. Jia et al., Phys. Rev. Lett. 129, 243402 (2022).

Overview

Superfluid velocity field

Quantized vortices $\Longleftrightarrow$ Signature of superfluidity (4He II - atomic BECs)

Scalar complex order parameter $\Psi = |\Psi| e^{i\Phi}$.

The associated superfluid velocity is

$$\boldsymbol{v} = (\hbar/M)\boldsymbol{\nabla}\Phi.$$

The superfluid velocity field is irrotational ($\boldsymbol{\nabla} \times \boldsymbol{v} = 0$) except at singular points that represent quantized vortices.

Quantized circulation around the closed contour $\mathcal{C}$

$$\kappa = \oint_{\mathcal{C}} d\boldsymbol{l} \cdot \boldsymbol{v} = \frac{\hbar}{M} \oint_{\mathcal{C}} d\boldsymbol{l} \cdot \boldsymbol{\nabla}\Phi = \frac{2\pi\hbar}{M}\nu,$$

where the fundamental unit of circulation is h/M; ν is integer.

Sream Function

Low temperatures $\Longleftrightarrow$ no dissipation, classical hydrodynamic equations

Superfluid $\Rightarrow$ Ideal fluids:
Irrotational ($\boldsymbol{\nabla} \times \boldsymbol{v} = 0$) and incompressible ($\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$).

An alternative representation of the superfluid velocity field

$$\boldsymbol{v} = \frac{\hbar}{M}\,\hat{\boldsymbol{n}} \times \boldsymbol{\nabla}\chi,$$

where $\chi(\boldsymbol{r})$ is the stream function and $\hat{\boldsymbol{n}} = \hat{x} \times \hat{y}$.

For an unbounded thin planar 2D superfluid film, the quantization of circulations means that for a single quantized vortex, the vorticity is singular, with

$$\boldsymbol{\nabla} \times \boldsymbol{v} = \frac{2\pi\hbar}{M}\,\hat{\boldsymbol{n}}\, q_0 \delta^{(2)}(\boldsymbol{r} - \boldsymbol{r}_0).$$

The scalar function χ satisfies Poisson's equation

$$\nabla^2\chi = 2\pi \sum_{j=1}^{N_v} q_j\, \delta^{(2)}(\boldsymbol{r} - \boldsymbol{r}_j),$$

with the N_v vortices at $\boldsymbol{r}_j$ and charges q_j.

Vortex in the complex plane

Velocity in terms of the potential fields

$$\boldsymbol{v} = (\hbar/M)\boldsymbol{\nabla}\Phi,$$

$$\boldsymbol{v} = (\hbar/M)\,\hat{\boldsymbol{n}} \times \boldsymbol{\nabla}\chi.$$

$\Longrightarrow$ Cauchy-Riemann equations for the complex potentials

Ideal fluids in 2D fully described by a complex potential

$$F(z) = \chi + i\,\Phi,$$

where $z = x + iy$, with $\mathrm{Re}\, F = \chi$ and $\mathrm{Im}\, F = \Phi$.

Derivative of F determines the hydrodynamic flow velocity

$$v_y + iv_x = \frac{\hbar}{M}\frac{dF}{dz} = \frac{\hbar}{M}F'(z).$$

Single vortex

The complex potential for a single vortex with charge q_0 at z_0 is

$$F(z) = q_0 \ln(z - z_0).$$

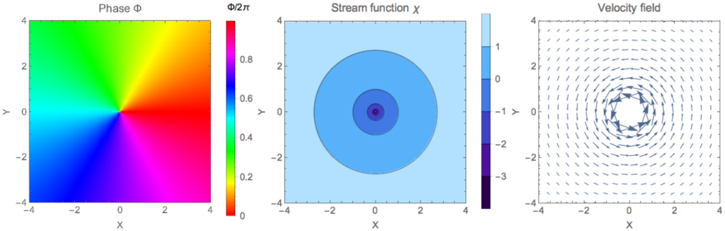
The resulting superfluid velocity $\boldsymbol{v}(\boldsymbol{r})$

$$v_y + iv_x = q_0 \frac{\hbar}{M} \frac{1}{z - z_0}.$$

If the vortex is at the origin $(z_0 = 0, q_0 = 1)$, we recover irrotational flow

$$\boldsymbol{v}_0(\boldsymbol{r}) = \frac{\hbar}{Mr} \hat{\phi},$$

where (r, ϕ) are plane polar coordinates and $\hat{\phi} = \hat{\boldsymbol{n}} \times \hat{\boldsymbol{r}}$.



Vortex dynamics

For a system of N_v vortices at $\boldsymbol{r}_j$ with vortex charge q_j ($j = 1, \cdots, N_v$), the total stream function $\chi(\boldsymbol{r})$ is the linear combination

$$\chi(\boldsymbol{r}) = \sum_{j=1}^{N_v} q_j \chi_j(\boldsymbol{r}),$$

with $\chi_j(\boldsymbol{r}) = \ln|\boldsymbol{r} - \boldsymbol{r}_j|$, giving the total hydrodynamic flow field

$$\boldsymbol{v}(\boldsymbol{r}) = \frac{\hbar}{M}\,\hat{\boldsymbol{n}} \times \boldsymbol{\nabla}\chi(\boldsymbol{r}) = \frac{\hbar}{M}\,\hat{\boldsymbol{n}} \times \boldsymbol{\nabla}\sum_j q_j \chi_j(\boldsymbol{r}).$$

In a ideal fluid, a given vortex moves with the local flow velocity at its position, typically arising from all the other vortices

$$\dot{\boldsymbol{r}}_k = \frac{\hbar}{M}\,\hat{\boldsymbol{n}} \times \boldsymbol{\nabla}_k {\sum_j}' q_j \chi_j(\boldsymbol{r}_k),$$

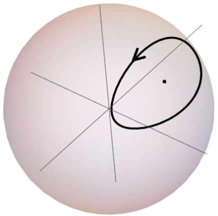
where we omitted the self-induced circulating flow.

Overview

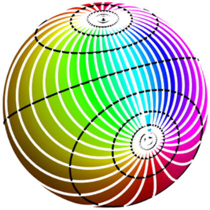
Superfluid on a sphere

Surperfluid film living on the surface of a sphere

- The compact topology requires the charge neutrality constraint $\sum_j q_j = 0$.



- Simple configuration: a single vortex dipole.

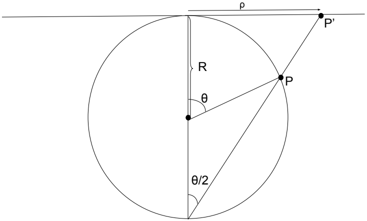


- Stereographic projection onto a complex plane to determine the stream function.

- Incompressible superfluid film:
- isotropic (s-wave) interaction: uniform thickness (δ) and density
- $\delta < \xi$ to avoid the line bending (point-vortex model - ξ_0)

Stereographic projection

In the stereographic projection, we consider a tangent plane at the north pole. Each point on the sphere has a one-to-one correspondence with a point on tangent plane, except for the point at the south pole.



A point on the sphere with spherical polar coordinates (θ, ϕ) has the corresponding complex coordinate on the plane

$$z = \rho e^{i\phi}$$

with

$$\rho = 2R \tan(\theta/2)$$

and the same azimuthal angle ϕ.

- Coordinate transformation to determine the velocity field on the sphere.

Dipole potential in the Sphere

For a vortex dipole with complex coordinates $z_\pm$ and charges $q_\pm = \pm 1$ we have

$$F_{\rm dip}(z) = \ln(z - z_+) - \ln(z - z_-) = \ln\left(\frac{z - z_+}{z - z_-}\right),$$

where z is on the tangent plane.
The stereographic projection gives the transformation $z_\pm = 2R\tan(\theta_\pm/2)e^{i\phi_\pm}$ and leads to the associated complex function on the sphere

$$F_{\rm dip}(\theta,\phi) = \ln\left(\frac{\tan(\theta/2)e^{i\phi} - \tan(\theta_+/2)e^{i\phi_+}}{\tan(\theta/2)e^{i\phi} - \tan(\theta_-/2)e^{i\phi_-}}\right).$$

Taking the real part, we obtain the stream function

$$\chi_{\rm dip}(\Omega) = \chi_+(\Omega) - \chi_-(\Omega).$$

with

$$\chi_j(\Omega) = \ln\left[2\sin(|\gamma_j|/2)\right],$$

with $\cos\gamma_j = \hat{\mathbf{r}}\cdot\hat{\mathbf{r}}_j$ and the chordal distance $d = 2R\sin(|\gamma_j|/2)$.

Dynamics of a Vortex dipole

The dipole stream function

$$\chi_-(\Omega_+) = \ln\left[2\sin\left(\frac{\theta_- - \theta_+}{2}\right)\right],$$

rotational symmetry: vortices at the same azimuth angle $\phi_+ = \phi_-$.

$\Rightarrow$ Velocity potential: $\boldsymbol{v}(\boldsymbol{r}) = (\hbar/M)\,\hat{\boldsymbol{r}} \times \boldsymbol{\nabla}\chi(\Omega)$

$$\dot{\boldsymbol{r}}_+ = \dot{\boldsymbol{r}}_- = \frac{\hbar}{2MR}\cot\left(\frac{\theta_- - \theta_+}{2}\right)\hat{\boldsymbol{\phi}} = \frac{\hbar}{2MR}\tan\theta\,\hat{\boldsymbol{\phi}},$$

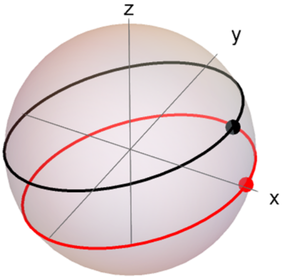
we set $\theta_+ = \theta$ and $\theta_- = \pi - \theta$.

Dynamics of a Vortex Dipole

- Each member of the vortex dipole moves uniformly in the positive $\hat{\phi}$ direction with fixed separation and speed $v_{\mathrm{dip}} \propto \tan\theta$.
- For small $\theta \ll 1$, the dipole is near the poles and the motion is slow.
- Near the equator ($\theta = \pi/2 - \Delta\theta$, with $\Delta\theta \ll 1$),

$$v_{\mathrm{dip}} \approx \frac{\hbar}{2MR\Delta\theta}$$

which corresponds to a planar dipole with linear separation $2R\Delta\theta$.



$\Rightarrow$ constant angular separation (constant chordal distance), parallel to the great circle that bisects the vortices.

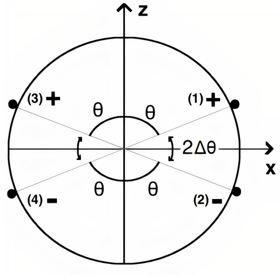
SYMMETRIC CONFIGURATION $++--$

All four vortices initially in the xz plane.

$\rightarrow$ For each $k = 1, \cdots, 4$

$$\dot{\boldsymbol{r}}_k = \frac{\hbar}{MR} \frac{1}{\sin 2\Delta\theta} \hat{\phi}_k,$$

where $\Delta\theta = \pi/2 - \theta$.



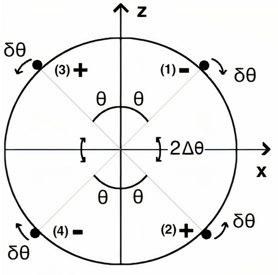
The vortices all rotate together in a positive (counterclockwise) sense around $\hat{\boldsymbol{z}}$, remaining coplanar.
coplanar-symmetric

- Near the poles ($\theta \ll 1$): pairs of same-sign vortices with speed $\hbar/(2MR\theta)$.
- Near the equator ($\theta \rightarrow \pi/2$ with $\Delta\theta \ll 1$): vortices $(1,2)$ and $(3,4)$ act like two independent vortex dipoles with speed $\hbar/(2MR\Delta\theta)$.

- For intermediate configurations, all four vortices influence each other, requiring a more detailed analysis.

EXCHANGED CONFIGURATION $+-+-$

Interchange the charge of the vortices 1 and 2, with $q_1 = -1$ and $q_2 = +1$.



$\rightarrow$ Initial configuration $\theta = \pi/4$

- Static configuration of the four vortices.

$$\dot{\boldsymbol{r}}_k = 0$$

$\rightarrow$ For small deviations of the initial positions with $\theta = \pi/4 + \delta\theta$

- Vortex executes a closed elliptical orbit around its equilibrium static configuration.

small-oscillation

Coplanar 4-vortices exchanged configuration

$\Rightarrow$ For small $\delta\theta$ we found that the analytical formula

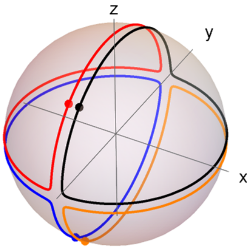
$$\omega \approx \frac{\hbar}{MR^2}\frac{1}{\cos(2\delta\theta)},$$

provides a best fit to the numerical values of the orbital frequency.

- For $\delta\theta \ll 1$, this frequency reduces to $\omega \approx \hbar/(MR^2)$.
- For small $\theta \ll 1$ $(= \delta\theta \to -\pi/4)$

$$\omega = \frac{2\pi}{T} = \frac{\hbar}{2MR^2\theta},$$

as expected for the orbital frequency of the dipoles with speed $\hbar/(2MR\theta)$ and travels a distance $2\pi R$ in one complete cycle.



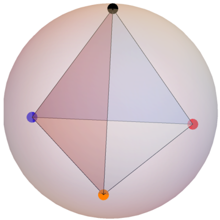
Tetrahedral configuration

⇒ Vortices at four symmetric sites on the sphere

- one possible configuration

- stationary condition

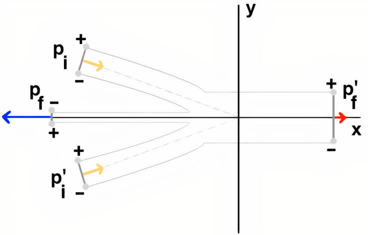
- unstable equilibrium (complicated dynamics even for small perturbations).



- For more general asymmetric θ configurations of the four vortices we are unable to predict their orbits analytically.

Dipole collision

$\Rightarrow$ Dynamics of two small vortex dipoles (dipole-collision)



- For small angular separation we define vortex dipole moment ($\boldsymbol{p}$)

- Energy conservation

$$E_{\text{tot}} = -(\pi\hbar^2 n/M)\ln(R^2/p_i p_i') \rightarrow p_i p_i' = p_f p_f'$$

- Momentum conservation

$$\boldsymbol{p}_i + \boldsymbol{p}_i' = \boldsymbol{p}_f + \boldsymbol{p}_f'$$

Overview

Vortex Dynamics on General Compact Surfaces

S. J. Bereta, M. A. Caracanhas, and A. L. Fetter, Phys. Rev. A 103, 053306 (2021).

- Bubble trap: effective axisymmetric ellipsoidal shell potential for the atoms, whose aspect ratio is determined by the quadrupole trap.
- How does the transition from a spherical shell to a spheroidal one affect the dynamics of a vortex dipole?
  - Nonuniform Gaussian curvature (K).

M. A. Caracanhas, P. Massignan, and A. L. Fetter, Phys. Rev. A 105, 023307 (2022).

- Conformal projection of an ellipsoidal surface onto the plane using isothermal coordinates.

Isothermal coordinates

- Metric in the axisymmetric compact surface (not isotropic !)

$$ds^2 = h_\phi^2 \left(\frac{h_\theta^2}{h_\phi^2} d\theta^2 + d\phi^2 \right),$$

where h_j are the metric parameters.

- Metric in the complex plane

$$ds^2 = \lambda^2 (d\rho^2 + \rho^2 d\phi^2),$$

with the overall scale factor λ.

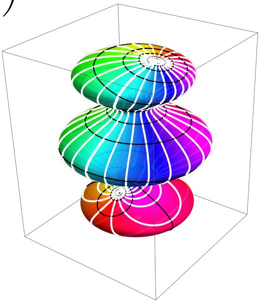
- Coordinate transformation to new "isothermal" variables (u, v), with isotropic metric $ds^2 = \lambda^2(du^2 + dv^2)$:

$$\ln \rho(\theta) = \int d\theta \, \frac{h_\theta}{h_\phi} \qquad \lambda(\theta) = \frac{h_\phi(\theta)}{\rho(\theta)}$$

Dipole potential on an axisymmetric compact surface

For a plane, the complex potential for a vortex dipole is the difference of two logarithms $F_{\rm dipole}(z) = \ln(z - z_+) - \ln(z - z_-) = \ln[(z - z_+)/(z - z_-)]$. The previous coordinate transformation now gives the desired expression for a vortex dipole on a quite general axisymmetric compact surface

$$F_{\rm dipole} = \ln\left(\frac{\rho\, e^{i\phi} - \rho_+\, e^{i\phi_+}}{\rho\, e^{i\phi} - \rho_-\, e^{i\phi_-}}\right)$$



- lines of constant phase

$$\Phi_{\rm dipole} = {\rm Im}\, F_{\rm dipole}$$

- lines of constant stream function

$$\chi_{\rm dipole} = {\rm Re}\, F_{\rm dipole}$$

Dipole dynamics on an axisymetric compact surface

The transformation from the curved surface to the complex plane introduces an additional contribution.

- On the physical surface $\Rightarrow$ same core size ξ_0 (regularize the singularity)
- On the complex plane $\Rightarrow$ $\xi_j = \xi_0/\lambda_j$

Renormalization includes a local curvature correction

$$\dot{\boldsymbol{r}}_+ = -\frac{\hbar}{M}\,\hat{\boldsymbol{n}}_+ \times \boldsymbol{\nabla}_+ \left(\chi_{+-} + \frac{1}{2}\ln\lambda_+\right).$$

These dynamical equations are readily generalized for an overall charge-neutral set of vortices

$$\dot{\boldsymbol{r}}_j = \frac{\hbar}{M}\,\hat{\boldsymbol{n}}_j \times \boldsymbol{\nabla}_j \left(\sum_k{}' q_k\,\chi_{jk} - \frac{1}{2} q_j\,\ln\lambda_j\right).$$

Total Energy - Hamiltonian equations

Total energy written in terms of the stream function χ_{kl} and the scale factor λ_k:

$$E = \frac{1}{2} M n \int d^2 r |\boldsymbol{v}(\boldsymbol{r})|^2$$
$$= \frac{\hbar^2 n \pi}{M} \left(-\sum_{kl}{}' q_k q_l \, \chi_{kl} + \sum_k q_k^2 \ln \lambda_k \right),$$

The partial derivatives of E yield the desired Hamiltonian equations of motion

$$2\pi\hbar \, n \, q_j \, \dot{\theta}_j = \frac{1}{h_{\theta_j} h_{\phi_j}} \frac{\partial E}{\partial \phi_j},$$
$$2\pi\hbar \, n \, q_j \, \dot{\phi}_j = -\frac{1}{h_{\theta_j} h_{\phi_j}} \frac{\partial E}{\partial \theta_j}.$$

E as the effective Hamiltonian and (θ_j, ϕ_j) as canonical variables.
$\Rightarrow dE/dt = 0$ (vortex dynamics conserves the total energy)

$$ds^2 = a^2 \left(h_\theta^2 \, d\theta^2 + h_\phi^2 \, d\phi^2 \right).$$

We now have *dimensionless* metric factors

$$h_\theta = \sqrt{\cos^2\theta + (b^2/a^2)\sin^2\theta}, \quad h_\phi = \sin\theta.$$

The Gaussian curvature K of an axisymmetric ellipsoid is

$$K = \frac{b^2}{\left(a^2\cos^2\theta + b^2\sin^2\theta\right)^2}$$

Gaussian curvature $K \Longleftrightarrow$ curvature contribution $\ln\lambda$ to the energy E.

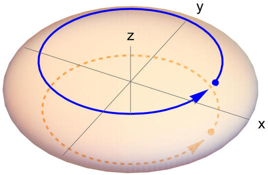
Brioschi's formula

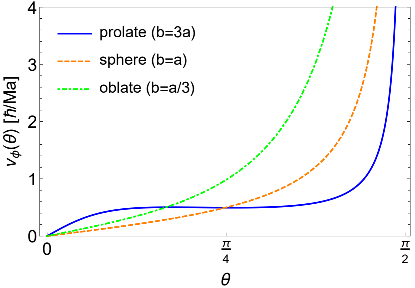
$$-\frac{1}{\lambda^2}\nabla^2 \ln\lambda = K,$$

a nonlinear Poisson equation for $\ln\lambda$ with the Gaussian curvature K as the source.

Vortex Dynamics on an ellipsoid

Dipole placed symmetrically around the equator

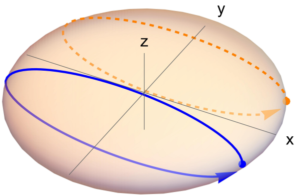




Circulating orbits of positive (solid blue) and negative (dashed orange) member of vortex dipole placed symmetrically above and below the equator.

The energy of a vortex dipole on an elongated surface depends linearly on the separation $d = |\boldsymbol{r}_+ - \boldsymbol{r}_-|$ once d exceeds the radial dimension, instead of the usual logarithmic dependence when d is small.

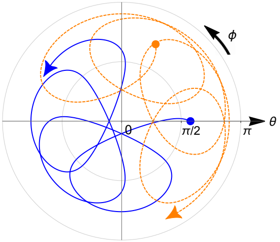
Vortex dipole moving north from equator



Circulating orbits of positive (solid blue) and negative (dashed orange) member of vortex dipole, starting from equator $\theta = \pi/2$, marked by dots.
- The dynamical motion indeed conserves the chordal distance.

General Asymmetric Case

If the initial positions of the two vortices do not satisfy any special symmetry, the ensuing dynamics becomes significantly more complex.

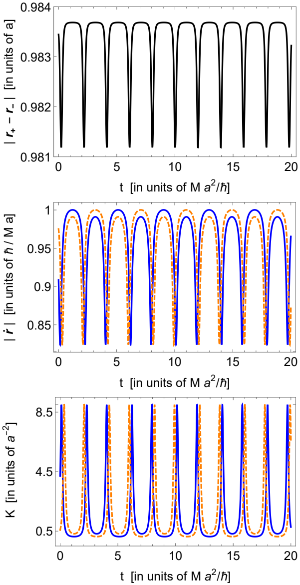


asymmetric-orbits

- Quasi-periodic motion.
- No closed orbits.

The dynamics conserves both the total energy E and the generalized angular momentum. The chordal distance between the vortices is not conserved.

General asymmetric case



- The chordal distance oscillates anharmonically but nonetheless periodically.
- The speed of each vortex also varies periodically but at half the frequency of the chordal distance.
- The local Gaussian curvature at the position of each vortex:

$$K = \frac{1-\epsilon_o^2}{a^2(1-\epsilon_o^2\sin^2\theta)^2},$$

- Anti-correlation between the vortex speed and the local curvature of the surface $\Rightarrow$ vortices slow down in the regions of higher curvature!

Overview

Conclusions and Outlook

• We presented a method to determine the dynamic of arbitrary single charged vortex-antivortex pairs configuration in general axisymmetric compact surfaces. Dynamic for vortex pairs in spheroidal surfaces - "Bubble" BECs.

• The results show that curvature can strongly influence vortex motion and may provide new mechanisms for controlling vortex dynamics in shell-shaped condensates.

• Dipolar condensates in the shell traps.

• Vortex dipoles on non-trivial compact geometries (toroids - quantized circulation loops).

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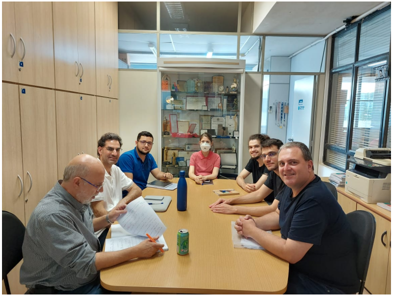

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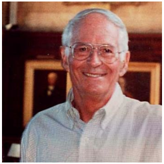

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Thank you!

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